EXAMPLE 6-50

▶ Suppose that the random variables x and y are $N(0, 0, \sigma_1^2, \sigma_2^2, r)$. As we know

$$E\{\mathbf{x}^2\} = \sigma_1^2 \qquad E\{\mathbf{x}^4\} = 3\sigma_1^4$$

Furthermore, $f(y \mid x)$ is a normal density with mean $r\sigma_2 x/\sigma_1$ and variance $\sigma_2\sqrt{1-r^2}$. Hence

$$E\{\mathbf{y}^2 \mid x\} = \eta_{y|x}^2 + \sigma_{y|x}^2 = \left(\frac{r\sigma_2 x}{\sigma_1}\right)^2 + \sigma_2^2(1-r^2) \tag{6-245}$$

Using (6-244), we shall show that

$$E\{\mathbf{xy}\} = r\sigma_1\sigma_2 \qquad E\{\mathbf{x}^2\mathbf{y}^2\} = E\{\mathbf{x}^2\}E\{\mathbf{y}^2\} + 2E^2\{\mathbf{xy}\} \tag{6-246}$$

Proof.

$$E\{\mathbf{xy}\} = E\{\mathbf{x}E\{\mathbf{y} \mid \mathbf{x}\}\} = E\left\{r\sigma_2\frac{\mathbf{x}^2}{\sigma_1}\right\} = r\sigma_2\frac{\sigma_1^2}{\sigma_1}$$

$$E\{\mathbf{x}^2\mathbf{y}^2\} = E\{\mathbf{x}^2 E\{\mathbf{y}^2 \mid \mathbf{x}\}\} = E\left\{\mathbf{x}^2\left[r^2\sigma_2^2\frac{\mathbf{x}^2}{\sigma_1^2} + \sigma_2^2(1-r^2)\right]\right\}$$

$$= 3\sigma_1^4 r^2\frac{\sigma_2^2}{\sigma_1^2} + \sigma_1^2\sigma_2^2(1-r^2) = \sigma_1^2\sigma_2^2 + 2r^2\sigma_1^2\sigma_2^2$$

and the proof is complete [see also (6-199)]. ◀

PROBLEMS

6-1 x and y are independent, identically distributed (i.i.d.) random variables with common p.d.f.

$$f_x(x) = e^{-x}U(x) \qquad f_y(y) = e^{-y}U(y)$$

Find the p.d.f. of the following random variables (*a*) $\mathbf{x}+\mathbf{y}$, (*b*) $\mathbf{x}-\mathbf{y}$, (*c*) $\mathbf{xy}$, (*d*) $\mathbf{x}/\mathbf{y}$, (*e*) $\min(\mathbf{x}, \mathbf{y})$, (*f*) $\max(\mathbf{x}, \mathbf{y})$, (*g*) $\min(\mathbf{x}, \mathbf{y})/\max(\mathbf{x}, \mathbf{y})$.

6-2 x and y are independent and uniform in the interval $(0, a)$. Find the p.d.f. of (*a*) $\mathbf{x}/\mathbf{y}$, (*b*) $\mathbf{y}/(\mathbf{x}+\mathbf{y})$, (*c*) $|\mathbf{x}-\mathbf{y}|$.

6-3 The joint p.d.f. of the random variables x and y is given by

$$f_{xy}(x, y) = \begin{cases} 1 & \text{in the shaded area} \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{z} = \mathbf{x}+\mathbf{y}$. Find $F_z(z)$ and $f_z(z)$.

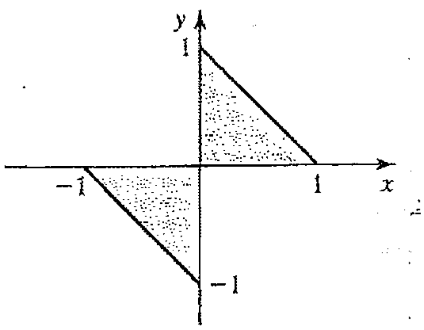

FIGURE P6-3

6-4 The joint p.d.f. of x and y is defined as

$$f_{xy}(x, y) = \begin{cases} 6x & x \geq 0, y \geq 0, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Find the p.d.f. of z.

6-5 x and y are independent identically distributed normal random variables with zero mean and common variance σ^2, that is, $\mathbf{x} \sim N(0, \sigma^2)$, $\mathbf{y} \sim N(0, \sigma^2)$ and $f_{xy}(x, y) = f_x(x) f_y(y)$. Find the p.d.f. of (*a*) $\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$, (*b*) $\mathbf{w} = \mathbf{x}^2 + \mathbf{y}^2$, (*c*) $\mathbf{u} = \mathbf{x} - \mathbf{y}$.

6-6 The joint p.d.f. of x and y is given by

$$f_{xy}(x, y) = \begin{cases} 2(1 - x) & 0 < x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability density function of $\mathbf{z} = \mathbf{xy}$.

6-7 Given

$$f_{xy}(x, y) = \begin{cases} x + y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that (*a*) $\mathbf{x} + \mathbf{y}$ has density $f_1(z) = z^2$, $0 < z < 1$, $f_1(z) = z(2 - z)$, $1 < z < 2$, and 0 elsewhere. (*b*) xy has density $f_2(z) = 2(1 - z)$, $0 < z < 1$, and 0 elsewhere. (*c*) y/x has density $f_3(z) = (1 + z)/3$, $0 < z < 1$, $f_3(z) = (1 + z)/3z^3$, $z > 1$, and 0 elsewhere. (*d*) $\mathbf{y} - \mathbf{x}$ has density $f_4(z) = 1 - |z|$, $|z| < 1$, and 0 elsewhere.

6-8 Suppose x and y have joint density

$$f_{xy}(x, y) = \begin{cases} 1 & 0 \leq x \leq 2, 0 \leq y \leq 1, 2y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Show that $\mathbf{z} = \mathbf{x} + \mathbf{y}$ has density

$$f_{xy}(x, y) = \begin{cases} (1/3)z & 0 < z < 2 \\ 2 - (2/3)z & 2 < z < 3 \\ 0 & \text{elsewhere} \end{cases}$$

6-9 x and y are uniformly distributed on the triangular region $0 \leq y \leq x \leq 1$. Show that (*a*) $\mathbf{z} = \mathbf{x}/\mathbf{y}$ has density $f_z(z) = 1/z^2$, $z \geq 1$, and $f_z(z) = 0$, otherwise. (*b*) Determine the density of xy.

6-10 x and y are uniformly distributed on the triangular region $0 < x \leq y \leq x + y \leq 2$. Find the p.d.f. of $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$.

6-11 x and y are independent Gamma random variables with common parameters α and β. Find the p.d.f. of (*a*) $\mathbf{x} + \mathbf{y}$, (*b*) $\mathbf{x}/\mathbf{y}$, (*c*) $\mathbf{x}/(\mathbf{x} + \mathbf{y})$.

6-12 x and y are independent uniformly distributed random variables on $(0, a)$. Find the joint p.d.f. of $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$.

6-13 x and y are independent Rayleigh random variables with common parameter σ^2. Determine the density of $\mathbf{x}/\mathbf{y}$.

6-14 The random variables x and y are independent and $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Find $f_y(y)$ if

$$f_x(x) = ce^{-cx} U(x) \qquad f_z(z) = c^2 z e^{-cz} U(z)$$

6-15 The random variables x and y are independent and y is uniform in the interval (0, 1). Show that, if $\mathbf{z} = \mathbf{x} + \mathbf{y}$, then

$$f_z(z) = F_x(z) - F_x(z - 1)$$

6-16 (*a*) The function $g(x)$ is monotone increasing and $\mathbf{y} = g(\mathbf{x})$. Show that

$$F_{xy}(x, y) = \begin{cases} F_x(x) & \text{if } \; y > g(x) \\ F_y(y) & \text{if } \; y < g(x) \end{cases}$$

(*b*) Find $F_{xy}(x, y)$ if $g(x)$ is monotone decreasing.

6-17 The random variables $\mathbf{x}$ and $\mathbf{y}$ are $N(0, 4)$ and independent. Find $f_z(z)$ and $F_z(z)$ if (*a*) $\mathbf{z} = 2\mathbf{x} + 3\mathbf{y}$, and (*b*) $\mathbf{z} = \mathbf{x}/\mathbf{y}$.

6-18 The random variables $\mathbf{x}$ and $\mathbf{y}$ are independent with

$$f_x(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} U(x) \qquad f_y(y) = \begin{cases} 1/\pi\sqrt{1-y^2} & |y| < 1 \\ 0 & |y| > 1 \end{cases}$$

Show that the random variable $\mathbf{z} = \mathbf{xy}$ is $N(0, \alpha^2)$.

6-19 The random variables $\mathbf{x}$ and $\mathbf{y}$ are independent with Rayleigh densities

$$f_x(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} U(x) \qquad f_y(y) = \frac{y}{\beta^2} e^{-y^2/2\beta^2} U(y)$$

(*a*) Show that if $\mathbf{z} = \mathbf{x}/\mathbf{y}$, then

$$f_z(z) = \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} U(z) \qquad \text{(i)}$$

(*b*) Using (i), show that for any $k > 0$,

$$P\{\mathbf{x} \le k\mathbf{y}\} = \frac{k^2}{k^2 + \alpha^2/\beta^2}$$

6-20 The random variables $\mathbf{x}$ and $\mathbf{y}$ are independent with exponential densities

$$f_x(x) = \alpha e^{-\alpha x} U(x) \qquad f_y(y) = \beta e^{-\beta y} U(y)$$

Find the densities of the following random variables:

(*a*) $2\mathbf{x} + \mathbf{y}$ (*b*) $\mathbf{x} - \mathbf{y}$ (*c*) $\dfrac{\mathbf{x}}{\mathbf{y}}$ (*d*) $\max(\mathbf{x}, \mathbf{y})$ (*e*) $\min(\mathbf{x}, \mathbf{y})$

6-21 The random variables $\mathbf{x}$ and $\mathbf{y}$ are independent and each is uniform in the interval $(0, a)$. Find the density of the random variable $\mathbf{z} = |\mathbf{x} - \mathbf{y}|$.

6-22 Show that (*a*) the convolution of two normal densities is a normal density, and (*b*) the convolution of two Cauchy densities is a Cauchy density.

6-23 The random variables $\mathbf{x}$ and $\mathbf{y}$ are independent with respective densities $\chi^2(m)$ and $\chi^2(n)$. Show that if (Example 6-29)

$$\mathbf{z} = \frac{\mathbf{x}/m}{\mathbf{y}/n} \qquad \text{then} \quad f_z(z) = \gamma \frac{z^{m/2-1}}{\sqrt{(1 + mz/n)^{m+n}}} U(z)$$

This distribution is denoted by $F(m, n)$ and is called the *Snedecor F* distribution. It is used in hypothesis testing (see Prob. 8-34).

6-24 Express $F_{zw}(z, w)$ in terms of $F_{xy}(x, y)$ if $\mathbf{z} = \max(\mathbf{x}, \mathbf{y})$, $\mathbf{w} = \min(\mathbf{x}, \mathbf{y})$.

6-25 Let $\mathbf{x}$ be the lifetime of a certain electric bulb, and $\mathbf{y}$ that of its replacement after the failure of the first bulb. Suppose $\mathbf{x}$ and $\mathbf{y}$ are independent with common exponential density function with parameter λ. Find the probability that the combined lifetime exceeds 2λ. What is the probability that the replacement outlasts the original component by λ?

6-26 $\mathbf{x}$ and $\mathbf{y}$ are independent uniformly distributed random variables in $(0, 1)$. Let

$$\mathbf{w} = \max(\mathbf{x}, \mathbf{y}) \qquad \mathbf{z} = \min(\mathbf{x}, \mathbf{y})$$

Find the p.d.f. of (*a*) $\mathbf{r} = \mathbf{w} - \mathbf{z}$, (*b*) $\mathbf{s} = \mathbf{w} + \mathbf{z}$.

6-27 Let x and y be independent identically distributed exponential random variables with common parameter λ. Find the p.d.f.s of (*a*) $\mathbf{z} = \mathbf{y}/\max(\mathbf{x}, \mathbf{y})$. (*b*) $\mathbf{w} = \mathbf{x}/\min(\mathbf{x}, 2\mathbf{y})$.

6-28 If x and y are independent exponential random variables with common parameter λ, show that $\mathbf{x}/(\mathbf{x}+\mathbf{y})$ is a uniformly distributed random variable in (0, 1).

6-29 x and y are independent exponential random variables with common parameter λ. Show that

$$\mathbf{z} = \min(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \mathbf{w} = \max(\mathbf{x}, \mathbf{y}) - \min(\mathbf{x}, \mathbf{y})$$

are independent random variables.

6-30 Let x and y be independent random variables with common p.d.f. $f_x(x) = \beta^{-\alpha}\alpha x^{\alpha-1}$, $0 < x < \beta$, and zero otherwise ($\alpha \geq 1$). Let $\mathbf{z} = \min(\mathbf{x}, \mathbf{y})$ and $\mathbf{w} = \max(\mathbf{x}, \mathbf{y})$. (*a*) Find the p.d.f. of $\mathbf{x}+\mathbf{y}$. (*b*) Find the joint p.d.f. of z and w. (*c*) Show that $\mathbf{z}/\mathbf{w}$ and w are independent random variables.

6-31 Let x and y be independent gamma random variables with parameters (α_1, β) and (α_2, β), respectively. (*a*) Determine the p.d.f.s of the random variables $\mathbf{x}+\mathbf{y}$, $\mathbf{x}/\mathbf{y}$, and $\mathbf{x}/(\mathbf{x}+\mathbf{y})$. (*b*) Show that $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}/\mathbf{y}$ are independent random variables. (*c*) Show that $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}/(\mathbf{x}+\mathbf{y})$ are independent gamma and beta random variables, respectively. The converse to (*b*) due to Lukacs is also true. It states that with x and y representing nonnegative random variables, if $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}/\mathbf{y}$ are independent, then x and y are gamma random variables with common (second) parameter β.

6-32 Let x and y be independent normal random variables with zero mean and unit variances. (*a*) Find the p.d.f. of $\mathbf{x}/|\mathbf{y}|$ as well as that of $|\mathbf{x}|/|\mathbf{y}|$. (*b*) Let $\mathbf{u} = \mathbf{x}+\mathbf{y}$ and $\mathbf{v} = \mathbf{x}^2+\mathbf{y}^2$. Are u and v independent?

6-33 Let x and y be jointly normal random variables with parameters μ_x, μ_y, σ_x^2, σ_y^2, and r. Find a necessary and sufficient condition for $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ to be independent.

6-34 x and y are independent and identically distributed normal random variables with zero mean and variance σ^2. Define

$$\mathbf{u} = \frac{\mathbf{x}^2 - \mathbf{y}^2}{\sqrt{\mathbf{x}^2+\mathbf{y}^2}} \qquad \mathbf{v} = \frac{\mathbf{x}\mathbf{y}}{\sqrt{\mathbf{x}^2+\mathbf{y}^2}}$$

(*a*) Find the joint p.d.f. $f_{uv}(u, v)$ of the random variables u and v. (*b*) Show that u and v are independent normal random variables. (*c*) Show that $[(\mathbf{x}-\mathbf{y})^2 - 2\mathbf{y}^2]/\sqrt{\mathbf{x}^2+\mathbf{y}^2}$ is also a normal random variable. Thus nonlinear functions of normal random variables *can* lead to normal random variables! (This result is due to Shepp.)

6-35 Suppose z has an F distribution with (m, n) degrees of freedom. (*a*) Show that $1/\mathbf{z}$ also has an F distribution with (n, m) degrees of freedom. (*b*) Show that $m\mathbf{z}/(m\mathbf{z}+n)$ has a beta distribution.

6-36 Let the joint p.d.f. of x and y be given by

$$f_{xy}(x, y) = \begin{cases} e^{-x} & 0 < y \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

Define $\mathbf{z} = \mathbf{x}+\mathbf{y}$, $\mathbf{w} = \mathbf{x}-\mathbf{y}$. Find the joint p.d.f. of z and w. Show that z is an exponential random variable.

6-37 Let

$$f_{xy}(x, y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define $\mathbf{z} = \mathbf{x}+\mathbf{y}$, $\mathbf{w} = \mathbf{y}/\mathbf{x}$. Determine the joint p.d.f. of z and w. Are z and w independent random variables?

6-38 The random variables x and θ are independent and θ is uniform in the interval $(-\pi, \pi)$. Show that if $z = x\cos(wt + \theta)$, then

$$f_z(z) = \frac{1}{\pi}\int_{-\infty}^{-|z|} \frac{f_x(y)}{\sqrt{y^2 - z^2}}\,dy + \frac{1}{\pi}\int_{|z|}^{\infty} \frac{f_x(y)}{\sqrt{y^2 - z^2}}\,dy$$

6-39 The random variables x and y are independent, x is $N(0, \sigma^2)$, and y is uniform in the interval $(0, \pi)$. Show that if $z = x + a\cos y$, then

$$f_z(z) = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_0^{\pi} e^{-(z - a\cos y)^2/2\sigma^2}\,dy$$

6-40 The random variables x and y are of discrete type, independent, with $P\{x = n\} = a_n$, $P\{y = n\} = b_n$, $n = 0, 1, \ldots$. Show that, if $z = x + y$, then

$$P\{z = n\} = \sum_{k=0}^{n} a_k b_{n-k}, \qquad n = 0, 1, 2, \ldots$$

6-41 The random variable x is of discrete type taking the values x_n with $P\{x = x_n\} = p_n$ and the random variable y is of continuous type and independent of x. Show that if $z = x + y$ and $w = xy$, then

$$f_z(z) = \sum_n f_y(z - x_n)p_n \qquad f_w(w) = \sum_n \frac{1}{|x_n|} f_y\left(\frac{w}{x_n}\right) p_n$$

6-42 x and y are independent random variables with geometric p.m.f.

$$P\{x = k\} = pq^k \quad k = 0, 1, 2, \ldots \qquad P\{y = m\} = pq^m \quad m = 0, 1, 2, \ldots$$

Find the p.m.f. of (*a*) $x + y$ and (*b*) $x - y$.

6-43 Let x and y be independent identically distributed nonnegative discrete random variables with

$$P\{x = k\} = P\{y = k\} = p_k \qquad k = 0, 1, 2, \ldots$$

Suppose

$$P\{x = k \mid x + y = k\} = P\{x = k - 1 \mid x + y = k\} = \frac{1}{k+1} \qquad k \geq 0$$

Show that x and y are geometric random variables. (This result is due to Chatterji.)

6-44 x and y are independent, identically distributed binomial random variables with parameters n and p. Show that $z = x + y$ is also a binomial random variable. Find its parameters.

6-45 Let x and y be independent random variables with common p.m.f.

$$P(x = k) = pq^k \qquad k = 0, 1, 2, \ldots \qquad q = p - 1$$

(*a*) Show that min(x, y) and $x - y$ are independent random variables. (*b*) Show that $z = \min(x, y)$ and $w = \max(x, y) - \min(x, y)$ are independent random variables.

6-46 Let x and y be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Show that the conditional density function of x given $x + y$ is binomial.

6-47 The random variables x_1 and x_2 are jointly normal with zero mean. Show that their density can be written in the form

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{\Delta}}\exp\left\{-\frac{1}{2}XC^{-1}X^t\right\} \qquad C = \begin{bmatrix}\mu_{11} & \mu_{12}\\ \mu_{21} & \mu_{22}\end{bmatrix}$$

where $X: [x_1, x_2]$, $\mu_{ij} = E\{x_i x_j\}$, and $\Delta = \mu_{11}\mu_{22} - \mu_{12}^2$.

6-48 Show that if the random variables x and y are normal and independent, then

$$P\{\mathbf{xy} < 0\} = G\left(\frac{\eta_x}{\sigma_x}\right) + G\left(\frac{\eta_y}{\sigma_y}\right) - 2G\left(\frac{\eta_x}{\sigma_x}\right)G\left(\frac{\eta_y}{\sigma_y}\right)$$

6-49 The random variables x and y are $N(0; \sigma^2)$ and independent. Show that if $\mathbf{z} = |\mathbf{x} - \mathbf{y}|$, then $E\{\mathbf{z}\} = 2\sigma/\sqrt{\pi}$, $E\{\mathbf{z}^2\} = 2\sigma^2$.

6-50 Show that if x and y are two independent exponential random variables with $f_x(x) = e^{-x}U(x)$, $f_y(y) = e^{-y}U(y)$, and $\mathbf{z} = (\mathbf{x} - \mathbf{y})U(\mathbf{x} - \mathbf{y})$, then $E\{\mathbf{z}\} = 1/2$.

6-51 Show that for any x, y real or complex (*a*) $|E\{\mathbf{xy}\}|^2 \le E\{|\mathbf{x}|^2\}E\{|\mathbf{y}|^2\}$; (*b*) (*triangle inequality*) $\sqrt{E\{|\mathbf{x}+\mathbf{y}|^2\}} \le \sqrt{E\{|\mathbf{x}|^2\}} + \sqrt{E\{|\mathbf{y}|^2\}}$.

6-52 Show that, if the correlation coefficient $r_{xy} = 1$, then $\mathbf{y} = a\mathbf{x} + b$.

6-53 Show that, if $E\{\mathbf{x}^2\} = E\{\mathbf{y}^2\} = E\{\mathbf{xy}\}$, then $\mathbf{x} = \mathbf{y}$.

6-54 The random variable n is Poisson with parameter λ and the random variable x is independent of n. Show that, if $\mathbf{z} = \mathbf{nx}$ and

$$f_x(x) = \frac{\alpha}{\pi(\alpha^2 + x^2)} \qquad \text{then} \quad \Phi_z(\omega) = \exp\{\lambda e^{-\alpha|\omega|} - \lambda\}$$

6-55 Let x represent the number of successes and y the number of failures of n independent Bernoulli trials with p representing the probability of success in any one trial. Find the distribution of $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Show that $E\{\mathbf{z}\} = n(2p - 1)$, $\mathrm{Var}\{\mathbf{z}\} = 4np(1 - p)$.

6-56 x and y are zero mean independent random variables with variances σ_1^2 and σ_2^2, respectively, that is, $\mathbf{x} \sim N(0, \sigma_1^2)$, $\mathbf{y} \sim N(0, \sigma_2^2)$. Let

$$\mathbf{z} = a\mathbf{x} + b\mathbf{y} + c \qquad c \neq 0$$

(*a*) Find the characteristic function $\Phi_z(u)$ of z. (*b*) Using $\Phi_z(u)$ conclude that z is also a normal random variable. (*c*) Find the mean and variance of z.

6-57 Suppose the conditional distribution of x given $\mathbf{y} = n$ is binomial with parameters n and p_1. Further, y is a binomial random variable with parameters M and p_2. Show that the distribution of x is also binomial. Find its parameters.

6-58 The random variables x and y are jointly distributed over the region $0 < x < y < 1$ as

$$f_{xy}(x, y) = \begin{cases} kx & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

for some k. Determine k. Find the variances of x and y. What is the covariance between x and y?

6-59 x is a Poisson random variable with parameter λ and y is a normal random variable with mean μ and variance σ^2. Further x and y are given to be independent. (*a*) Find the joint characteristic function of x and y. (*b*) Define $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Find the characteristic function of z.

6-60 x and y are independent exponential random variables with common parameter λ. Find (*a*) $E[\min(\mathbf{x}, \mathbf{y})]$, (*b*) $E[\max(2\mathbf{x}, \mathbf{y})]$.

6-61 The joint p.d.f. of x and y is given by

$$f_{xy}(x, y) = \begin{cases} 6x & x > 0, y > 0, 0 < x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $\mathbf{z} = \mathbf{x} - \mathbf{y}$. (*a*) Find the p.d.f. of z. (*b*) Find the conditional p.d.f. of y given x. (*c*) Determine $\mathrm{Var}\{\mathbf{x} + \mathbf{y}\}$.

6-62 Suppose x represents the inverse of a chi-square random variable with one degree of freedom, and the conditional p.d.f. of y given x is $N(0, x)$. Show that y has a Cauchy distribution.

6-63 For any two random variables x and y, let $\sigma_x^2 = \text{Var}\{x\}$, $\sigma_y^2 = \text{Var}\{y\}$ and $\sigma_{x+y}^2 = \text{Var}\{x+y\}$.
(*a*) Show that

$$\frac{\sigma_{x+y}}{\sigma_x + \sigma_y} \le 1$$

(*b*) More generally, show that for $p \ge 1$

$$\frac{\{E(|x+y|^p)\}^{1/p}}{\{E(|x|^p)\}^{1/p} + \{E(|y|^p)\}^{1/p}} \le 1$$

6-64 x and y are jointly normal with parameters $N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho_{xy})$. Find (*a*) $E\{y \mid x = x\}$, and (*b*) $E\{x^2 \mid y = y\}$.

6-65 For any two random variables x and y with $E\{x^2\} < \infty$, show that (*a*) $\text{Var}\{x\} \ge E[\text{Var}\{x \mid y\}]$. (*b*) $\text{Var}\{x\} = \text{Var}[E\{x \mid y\}] + E[\text{Var}\{x \mid y\}]$.

6-66 Let x and y be independent random variables with variances σ_1^2 and σ_2^2, respectively. Consider the sum

$$z = ax + (1-a)y \qquad 0 \le a \le 1$$

Find a that minimizes the variance of z.

6-67 Show that, if the random variable x is of discrete type taking the values x_n with $P\{x = x_n\} = p_n$ and $z = g(x, y)$, then

$$E\{z\} = \sum_n E\{g(x_n, y)\}p_n \qquad f_z(z) = \sum_n f_z(z \mid x_n)p_n$$

6-68 Show that, if the random variables x and y are $N(0, 0, \sigma^2, \sigma^2, r)$, then

(*a*)
$$E\{f_y(y \mid x)\} = \frac{1}{\sigma\sqrt{2\pi(2-r^2)}} \exp\left\{-\frac{r^2x^2}{2\sigma^2(2-r^2)}\right\}$$

(*b*)
$$E\{f_x(x)f_y(y)\} = \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$$

6-69 Show that if the random variables x and y are $N(0, 0, \sigma_1^2, \sigma_2^2, r)$ then

$$E\{|xy|\} = \frac{2}{\pi}\int_0^c \arcsin\frac{\mu}{\sigma_1\sigma_2}\,d\mu + \frac{2\sigma_1\sigma_2}{\pi} = \frac{2\sigma_1\sigma_2}{\pi}(\cos\alpha + \alpha\sin\alpha)$$

where $r = \sin\alpha$ and $C = r\sigma_1\sigma_2$.
(*Hint:* Use (6-200) with $g(x, y) = |xy|$.)

6-70 The random variables x and y are $N(3, 4, 1, 4, 0.5)$. Find $f(y \mid x)$ and $f(x \mid y)$.

6-71 The random variables x and y are uniform in the interval $(-1, 1)$ and independent. Find the conditional density $f_r(r \mid M)$ of the random variable $r = \sqrt{x^2 + y^2}$, where $M = \{r \le 1\}$.

6-72 Show that, if the random variables x and y are independent and $z = x + y$, then $f_z(z \mid x) = f_y(z - x)$.

6-73 Show that, for any x and y, the random variables $z = F_x(x)$ and $w = F_y(y \mid x)$ are independent and each is uniform in the interval (0, 1).

6-74 We have a pile of m coins. The probability of heads of the ith coin equals p_i. We select at random one of the coins, we toss it n times and heads shows k times. Show that the probability that we selected the rth coin equals

$$\frac{p_r^k(1-p_r)^{n-k}}{p_1^k(1-p_1)^{n-k} + \cdots + p_m^k(1-p_m)^{n-k}}$$

6-75 The random variable x has a Student t distribution $t(n)$. Show that $E\{x^2\} = n/(n-2)$.

6-76 Show that if $\beta_x(t) = f_x(t \mid \mathbf{x} > t)$, $\beta_y(t \mid \mathbf{y} > t)$ and $\beta_x(t) = k\beta_y(t)$, then $1 - F_x(x)$ $[1 - F_y(x)]^k$.

6-77 Show that, for any $\mathbf{x}$, $\mathbf{y}$, and $\varepsilon > 0$,

$$P\{|\mathbf{x} - \mathbf{y}| > \varepsilon\} \le \frac{1}{\varepsilon^2} E\{|\mathbf{x} - \mathbf{y}|^2\}$$

6-78 Show that the random variables $\mathbf{x}$ and $\mathbf{y}$ are independent iff for any a and b:

$$E\{U(a - \mathbf{x})U(b - \mathbf{y})\} = E\{U(a - \mathbf{x})\}E\{U(b - \mathbf{y})\}$$

6-79 Show that

$$E\{\mathbf{y} \mid \mathbf{x} \le 0\} = \frac{1}{F_x(0)} \int_{-\infty}^{0} E\{\mathbf{y} \mid \mathbf{x}\} f_x(x)\, dx$$