EE 202 Lecture Notes

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Chapter 1

Nth Order Circuits

We examine the time-domain description of the Nth order linear time-invariant circuits. Our interest is the characterization of the dynamic circuits and the types of solutions with a particular focus on unforced (zero-input) solution. We revisit the concept of natural frequencies and examine the modes of circuit and give the details on the response calculation for circuits excited with initial conditions. Finally, we examine the concepts related to the stability of circuits in the absence of external forcing. The topics presented in this chapter are fundamental to the general linear system theory and are part of the fundamental knowledge base that every engineer practices.

The chapter starts with the discussion of linearity. Different aspects of linearity conditions for the dynamic circuits (systems) are discusses. Then the types of responses (particular, homogeneous etc.) and the methods of their calculation are given. At some parts, the chapter carries a little more detailed information than typically required required from a second year student. These details are presented to show the subtle intricacies that may possibly misguide or confuse the careful readers.

1.1 Operators, Systems, Linearity

An operator maps functions to functions. For the purposes of circuit theory, the operators can be considered to operate on the time functions. For example, a circuit with the external input of $\cos(2t + 45°)$ for $t \geq 0$ can *cause* a branch voltage of $1/2\cos(2t)$ Volts for $t \geq 0$. The mapping of the functions between the external input (external forcing)and the branch voltage can be interpreted as the action of an operator on the input of $\cos(2t + 45°)$. The conclusion is that the operator "modifies" or "reshapes" the given input function to another function. To implement the operators, we build systems, i.e. built circuits. Among all systems, the linear systems are the most fundamental and also the most suitable for analysis. Our discussion in this course is solely limited to linear and time-invariant systems.

A system $\mathcal{L}$ with the input $x(t)$ and the output $y(t)$, $\mathcal{L}\ \{x(t)\} = y(t)$, is called linear if the following conditions are satisfied:

1. Scaling: $\mathcal{L}\{\alpha x(t)\} = \alpha y(t), \quad \forall\alpha$
2. Superposition: $\mathcal{L}\{\alpha x_1(t) + \beta x_2(t)\} = \alpha y_1(t) + \beta y_2(t), \quad \forall\alpha, \forall\beta$
   where $\mathcal{L}\ \{x_1(t)\} = y_1(t)$ and $\mathcal{L}\ \{x_2(t)\} = y_2(t)$

It is important to note that the conditions should be satisfied for any $x(t)$ and $y(t)$, not for specially chosen ones.

It can be noted that when the input is the zero function $x(t) = 0$ (zero-input condition), then a linear system should a zero output. (To show this, you may take $x_2(t) = x_1(t)$ and $\alpha = 1, \beta = -1$ in the superposition property.) The simple zero-input and zero-output condition can be quick check for the linearity of the system. But note that, when the zero-input and zero-output check is satisfied; we can not say that the system is linear or not! If the zero-input and zero-output condition is not satisfied, we can surely say that the system is *not* linear. Hence the zero-input and zero-output condition is *not sufficient* to claim the linearity of the system; but it is a *necessary* condition to declare the linearity of the system.

Some important examples of linear operators, especially related to the circuit applications, are $D = \frac{d}{dt}$, $D^{-1} = \int_{-\infty}^{t}(\cdot)dt'$ and the multiplication by R operator. In addition, it is also easy to see that the cascade application of these linear operators also result in a linear operator; that is multiplication of $x(t)$ by R and then differentiation is also linear operator. (Make sure to show this result!)

Two linear operators $\mathcal{A}$ and $\mathcal{B}$ are said to commute if $\mathcal{A}\{\mathcal{B}\{x(t)\}\} = \mathcal{B}\{\mathcal{A}\{x(t)\}\}$. For example, if $\mathcal{A}$ is the multiplication by R operator and $\mathbf{B}$ is the time differentiation, then $\mathcal{A}\{\mathcal{B}\{x(t)\}\} = \mathcal{B}\{\mathcal{A}\{x(t)\}\}$, i.e. two operators commute. When two operators commute, this means that the order in their application does not matter. Hence, for this example it does not matter whether you multiply the function $x(t)$ by R first and differentiate the result or differentiate first and then multiply the result by R.

As a second example to the commutativity, lets assume that $\mathcal{A} = \frac{d}{dt}$ and $\mathcal{B} = \int_0^t(\cdot)dt'$. Then $\mathcal{A}\{\mathcal{B}\{x(t)\}\} = \frac{d}{dt}\{\int_0^t x(t')dt'\} = x(t)$, while $\mathcal{B}\{\mathcal{A}\{x(t)\}\} = \int_0^t\{\frac{d}{dt'}x(t')\}dt' = x(t) - x(0)$. Hence, these two operators does not commute unless $x(0) = 0$. If we change the lower limit of integration operator $\mathcal{B}$ from 0 to $-\infty$, that is $\mathcal{B} = \int_{-\infty}^{t} (\cdot)dt'$; and adopt the convention that $x(-\infty) = 0$ then $\frac{d}{dt}$ and $\int_{-\infty}^{t} (\cdot)dt'$ commute. In this chapter, we use the operator D to denote $\frac{d}{dt}$ and D^{-1} to denote $\int_{-\infty}^{t} (\cdot)dt'$ unless otherwise is explicitly stated. With this definition D and D^{-1} commute, then the complicated cascade application of these operators, such as $D^{-1}(D+1)(D+2)$, $(D+1)D^{-1}(D+2)$, $(D+1)(D+2)D^{-1}$ are identically the same. Essentially, with the adoption of the presented notation and convention; the operator D and its inverse D^{-1} can be treated as like a ordinary polynomial in D.

An operator description such as $y(t) = \left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) x(t)$ can be considered as the cascade of $(\frac{d}{dt} + 2)$ and $(\frac{d}{dt} + 1)$ operators. Here $x(t)$ is the input and $y(t)$ is the output of the system. In plain words, $y(t)$ can be described as the second derivative of $x(t)$ plus 3 times the first derivative of $x(t)$ and 2 times the function $x(t)$. It is easy to see that this operator is linear.

A more interesting question is the linearity of the following operator:

$$
\begin{aligned}
\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) y(t) &= f(t) \\
y(0) &= y_0 \\
y'(0) &= y'_0
\end{aligned}
\tag{1.1}
$$

Here $f(t)$ is the input (the forcing term) and $y(t)$ is the output of the system. We know that the initial conditions of $y(t)$ at $t = 0$. Hence we have a typical 2nd order constant coefficient differential equation.

Let's check whether the system given in (1.1) is linear or not. For linearity, if the input is zero function ($f(t) = 0$), the output $y(t)$ should be identical to zero. If this is not true, the system can not be linear. This shows that unless both initial conditions are zero, this system can not be linear. So, the initial conditions should be zero for linearity. (Going back to EE201, you may remember that we enforce the zero-state condition (all zero initial conditions) for the impulse and step response calculations. The zero-state responses, by their definition, satisfy the linearity conditions and this allows us to superpose many external inputs for the zero-state solution!) Let's assume zero initial conditions in (1.1):

$$
\begin{aligned}
\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) y(t) &= f(t) \\
y(0) &= 0 \\
y'(0) &= 0
\end{aligned}
\tag{1.2}
$$

The zero-initial conditions is not sufficient to guarantee the linearity. To check the linearity of the system in (1.2), we assume that the input $f(t)$ and causes an output $y(t)$, i.e. the output $y(t)$ is the zero-state solution to the input $f(t)$. If the input is changed to $\alpha f(t)$ (α is an arbitrary scalar), the response should be $\alpha y(t)$ for linearity. This is indeed true for the given equation. We can show this by multiplying both sides of the differential equation $\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)y(t)=x(t)$ by α. Once this is done, we get another differential equation whose input is $\alpha f(t)$. It is clear that $\alpha y(t)$ satisfies this differential equation. This is what we want to show! (Note that initial condition for the system with input $\alpha x(t)$ is all zero, so $\alpha x(t)$ also satisfies the initial condition.)

To check the superposition property of the linear systems, we multiply both sides of $\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)y_k(t) = f_k(t)$ by α and β to get $\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)\alpha y_1(t) = \alpha f_1(t)$ and $\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)\beta y_2(t) = \beta f_2(t)$. By adding these two equations we get $\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)\{\alpha y_1(t)+\beta y_2(t)\} = \alpha f_1(t)+\beta f_2(t)$. Hence it is shown that if input $f_1(t)$ causes $y_1(t)$ and $f_2(t)$ causes $y_2(t)$; then the input $\alpha f_1(t)+\beta f_2(t)$ results in $\alpha y_1(t)+\beta y_2(t)$ at the output. Again please note that, it initials conditions are not zero, the superposition principle is not satisfied. If $y_1(t)$ and $y_2(t)$ are not zero at $t=0$, then $\alpha y_1(0)+\beta y_2(0)$ can not be zero (for any α and β), hence the initial condition of the differential equation can not be satisfied with the solution of $\alpha y_1(t)+\beta y_2(t)$.

In the following section, we start the solution of dynamic circuits. Our goal is to characterize the mapping between the input and output of a linear time-invariant dynamic circuit. The input can be the initial conditions, external forcing or both.

1.2 Types of Responses

The solution of Nth order constant coefficient differential equations is accomplished in a few steps. The steps of solution carry important on their own. These steps towards the solution can be considered as the zero-input solution, the zero-state solution and their combination is the complete solution. We illustrate these solutions on the following example:

$$\begin{aligned}
\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)y(t) &= x(t)\\
y(0) &= y_0\\
y'(0) &= y'_0
\end{aligned}$$

1.2.1 Circuit Resposes

Zero-Input Response:

As the name implies, this is the response when the input is the zero function (hence $x(t) = 0$). This is the response solely due to the initial conditions.

$$\begin{aligned} \left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) y_{zi}(t) &= 0 \\ y(0) &= y_0 \\ y'(0) &= y_0' \end{aligned}$$

The solution of the given differential equation can be found as $y_{zi}(t) = c_1 e^{-t} + c_2 e^{-2t}$ for $t \geq 0$. Here c_1 and c_2 are arbitrary real numbers, i.e. for any c_1, c_2 the differential equation is satisfied. The zero-input solution is the unique solution satisfying both the differential equation (the signal evolution rule for $t > 0$) and the initial conditions. (You may also consider $\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)$ as an operator $\mathcal{A}\{\cdot\}$. Then the equation $\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) y_{zi}(t) = 0$, can be written as $\mathcal{A}\{y_{zi}(t)\} = 0$. That is, we are seeking the null space of the operator $\mathcal{A}$. The null space of $\mathcal{A}$ is spanned by e^{-t} and e^{-2t}, that is any function that can be written as a linear combination of e^{-t} and e^{-2t} is in the null space, $c_1 e^{-t} + c_2 e^{-2t}$. The solution we seek is the one in the null space satisfying the initial conditions.)

The initial conditions are found as follows:

$$\begin{aligned} y_{zi}(0) &= \left[c_1 e^{-t} + c_2 e^{-2t}\right]_{t=0} = c_1 + c_2 = y_0 \\ y_{zi}'(0) &= \left[-c_1 e^{-t} - 2c_2 e^{-2t}\right]_{t=0} = -c_1 - 2c_2 = y_o' \end{aligned}$$

From these two equations, we can get $c_1 = 2y_0 + y_0'$; $c_2 = -y_0' - y_0$ and the zero-input response (the special point in the null-space) is

$$y_{zi}(t) = (2y_0 + y_o')e^{-t} - (y_o' + y_0)e^{-2t}$$

Zero-state Response:

Zero-state response is the response due to the forcing term, (external excitation). It is assumed that the circuit or the system at rest, that is having zero initial conditions. Another way of expressing the same fact is that the system has no energy at $t = 0$.

$$\begin{aligned}\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)y_{zs}(t) &= f(t)\\ y_{zs}(0) &= 0\\ y_{zs}^{'}(0) &= 0\end{aligned}$$

The solution for the zero-state response depends on the input $f(t)$ and the initial conditions. As an example, assume that the input is unit step function, $f(t) = u(t)$, then the response is $y_{zs}(t) = \frac{1}{2} + c_1e^{-t} + c_2e^{-2t}$ for $t > 0$. This is solution satisfying the differential equation for $t > 0$. To meet the initial conditions, c_1, c_2 should be properly selected. If we do that, we get the solution satisfying both the differential relation and initial conditions as follows $y_{zs}(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$ for $t \geq 0$.

Note that the zero-state response obeys the linearity conditions. Because of this, we define the impulse response, the step response and other responses only for the zero-state condition. This is critical and it can be difficult to appreciate the importance of this at an initial introduction.

Complete Response:

Complete solution of the circuit which is the combination of both zero-input and zero-state responses.

$$y_{\text{comp}}(t) = y_{zi}(t) + y_{zs}(t)$$

For the example given, the complete solution for unit-step input is $y_{\text{comp}}(t) = \frac{1}{2} + (2y_0 + y_0' - 1)e^{-t} - (y_0' + y_0 - \frac{1}{2})e^{-2t}$.

1.2.2 Homogeneous and Particular Solutions

Homogeneous Solution:

This is the solution of the differential equation when forcing term is equal to zero.Homogeneous solution is similar to the zero-input response, but not the same. The difference between the homogeneous solution and the zero-input response is that the homogeneous solution contains undetermined coefficients typically shown as c_1 and c_2. While the zero-input solution, does not have any undetermined parameters; it is the solution of a circuit when the input is zero (as the name implies) and there is only initial conditions exciting the circuit. In other words, when you fix the undetermined parameters of the homogeneous solution to match a given set of initial conditions, you have the zero-input solution.

The following shows the distinction between homogeneous and zero-input solutions

$$\begin{aligned}\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)y_h(t) &= 0 \\ y^h(t) &= c_1e^{-t}+c_2e^{-2t} \\ y^{zi}(t) &= y^h(t)\downarrow_{c_1=2,\,c_2=-2} = 2e^{-t}-2e^{-2t}\end{aligned} \tag{1.3}$$

As shown in (1.3), the zero-input solution is formed by setting c_1 and c_2 to meet zero-initial conditions. In this example, they are arbitrarily selected to be 2 and -2.

From a more general perspective, $y_h(t)$ is not a response to a particular initial conditions; but it is a solution space which is the null space of the operator, $y_h(t) = c_1e^{-t}+c_2e^{-2t}$. The zero-input solution is a specific point in the null space. We select this specific point by setting c_1 and c_2.

Particular Solution:

This is the solution of the differential equation due to the forcing terms. This solution is similar to the zero-state solution, but not the same.

$$\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)y_p(t) = x(t) \tag{1.4}$$

For $x(t) = 1$, $y_p(t) = \frac{1}{2}$. The zero-state solution for this input is $y^{zs}(t) = \frac{1}{2} + c_1e^{-t}+c_2e^{-2t}$ where c_1 and c_2 should be selected to meet zero-initial conditions, i.e. all state variables are zero.

Complete Solution: This is the solution of the system, which is

$$y_{comp}(t) = y_h(t) + y_p(t) \tag{1.5}$$

Note that $y_h(t)$ contains undetermined c_1 and c_2 coefficients. These coefficients should be set to meet the given initial conditions.

The same complete solution can also be written as $y_{\text{comp}}(t) = y_{zi}(t) + y_{zs}(t)$. Note that $y_{zi}(t)$ and $y_{zs}(t)$ does not contain any undetermined coefficients.

1.3 Solution of Nth order Dynamic Circuits

We proceed towards the solution of Nth order dynamic circuits. In EE201, we have studied RC, RL and RLC circuits. These circuits form the special cases of first and second order circuits of the general Nth order circuits.

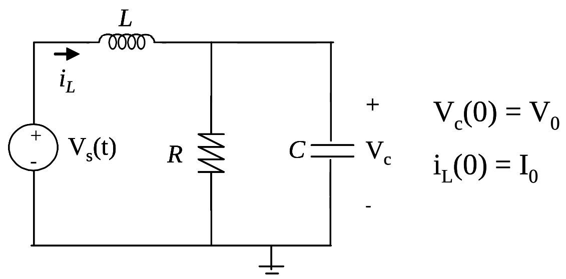

Figure 1.1: A 2nd order circuit for the illustration of the solution approach.

We present two analysis approaches. In the first approach, we use a scalar Nth order differential equation and find its solution. Then we repeat the analysis with the state equations. We would like to illustrate the equivalency of two techniques and emphasize the power of state-equation formalism in comparison to other methods. In this section, we examine the circuit given in Figure 1.1.

1.3.1 Solution by 2nd order scalar differential equation

The circuit contains two nodes (not counting datum) and a single voltage source, therefore the solution can be expressed through a single node equation. Then writing KCL at node whose voltage is $V_C(t)$, we get

$$C\dot{V}_c(t) + \frac{V_C(t)}{R} - \left(i_L(0) + \frac{1}{L}\int_0^t (V_s(\tau) - V_C(\tau))d\tau\right) = 0 \tag{1.6}$$

By taking the second derivative of the equation above and remembering the fundamental theorem of calculus, $\frac{d}{dt}\int_0^t x(\tau)d\tau = x(t)$, we get

$$\begin{aligned} C\ddot{V}_c(t) + \frac{\dot{V}_c(t)}{R} + \frac{V_C(t)}{L} &= \frac{V_s(t)}{L} \\ \left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) &= \frac{V_s(t)}{LC}. \end{aligned} \tag{1.7}$$

To solve the differential equation, we need two initial conditions. We can get these initial conditions at $t = 0^+$ through the analysis given in Figure 1.2.

From the circuit given in figure 1.2, we get the initial conditions for $V_C(0^+)$ and $V'c(0^+)$ as follows:

$$\begin{aligned} V_C(0^+) &= V_0 \\ V^{'}c(0^+) &= \frac{1}{C}\left(I_0 - \frac{V_0}{R}\right) \end{aligned} \tag{1.8}$$

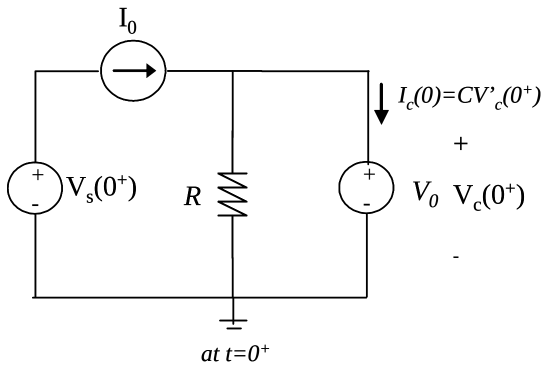

Figure 1.2: Circuit shown Figure1.1 at $t = 0^+$.

Now, we are ready to go and find the solution to the differential equation. We rewrite the equation below for convenience.

$$\begin{aligned} \left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) &= \frac{V_s(t)}{LC} \\ V_C(0^+) &= V_0 \\ V'c(0^+) &= \frac{1}{C}\left(I_0 - \frac{V_0}{R}\right) \end{aligned} \tag{1.9}$$

First, we focus on the zero-input response, that is

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C^{zi}(t) = 0 \tag{1.10}$$

$$V_C^{zi}(0^+) = V_0 \tag{1.11}$$

$$\frac{d}{dt}V_C^{zi}(0^+) = \frac{1}{C}\left(I_0 - \frac{V_0}{R}\right) \tag{1.12}$$

Then, we use the method of undetermined coefficients, that is we assume that

$$V_C^{zi}(t) = ce^{\lambda t} \tag{1.13}$$

This is a guess (an educated guess) about the form of the solution. We do not know c and λ yet and we hope to get them right so that we have the solution of the differential equation. Due to uniqueness of the solution for differential equations, if we can find a solution, it is *the* solution.

At this point we set some numerical values for R, L and C. Setting $R = 1/3$,$C = 1$ and $L = 1/2$, we get

$$\begin{aligned} \left(D^2 + 3D + 2\right) V_C(t) &= 2V_s(t) \\ V_C(0^+) &= V_0 \\ V'c(0^+) &= I_0 - 3V_0 \end{aligned} \tag{1.14}$$

By substituting $V_C^{zi}(t) = ce^{\lambda t}$ into $(D^2 + 3D + 2)\, V_C^{zi}(t) = 0$ we get,

$$\left(\lambda^2 + 3\lambda + 2\right) ce^{\lambda t} = 0$$

This equation has to be satisfied. Since $e^{\lambda t} > 0$ for all t and λ, we can divide both sides by $e^{\lambda t}$ and get

$$\left(\lambda^2 + 3\lambda + 2\right) c = 0 \tag{1.15}$$

To satisfy the last equation, we can set $c = 0$. This indeed guarantees the equality of right and left hand side of (1.15); but this is an uninteresting solution, $V_C^{zi}(t) = 0$. Furthermore it is not possible to satisfy the given non-zero initial conditions with this solution (this solution is called trivial solution). So we would better assume that $c \neq 0$ and look for another way satisfying (1.15). Since $c \neq 0$, we can dividing both sides of (1.15) by c to get

$$\left(\lambda^2 + 3\lambda + 2\right) = 0$$

The solution of this polynomial equation is $\lambda = \{-1, -2\}$. Then for $c \neq 0$ and $\lambda = \{-1, -2\}$ are two possible non-trivial solutions!

$$V_C^{zi}(t) = ce^{-t} + de^{-2t} \tag{1.16}$$

Each term in the zero-input solution is called a *mode* of the system. The given example has two modes, ce^{-t} and de^{-2t}. The modes have different decay rates. In this example, the second mode (the mode with $\lambda = -2$) decays two times faster than the other mode. First order RC and RL circuits have a single mode, their decay is $1/\tau$ where τ is the time-constant of the circuit.

Single Mode Excitation

In this section, we examine an interesting problem. The problem is the selection of initial conditions such that there is only mode of the circuit is excited. In other words, we would like to find a specific initial condition to excite a single mode of the circuit or equivalently, we would like to have solution in the form $V_C^{zi}(t) = ce^{-t}$ or $V_C^{zi}(t) = de^{-2t}$ for the example given in Figure 1.1.

If the solution is in the form, $V_C^{zi}(t) = ce^{-t}$; then at $t = 0^+$, we have $V_C^{zi}(0) = c$ and $\frac{d}{dt}V_C^{zi}(0) = -c$. From Figure 1.2, we can note the relation between initial conditions and the capacitor voltage and its derivates as follows:

$$\begin{aligned} V_C(0^+) &= V_0 \\ V_C'(0^+) &= I_0 - 3V_0 \end{aligned}$$

Finally, then by setting $I_0 = 2c, V_0 = c$, or $\begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, leads to a solution in the form $V_C^{zi}(t) = ce^{-t}$. So by setting the the initial voltage of capacitor half of the initial current of inductor, we can excite the mode with $\lambda = -1$.

Similarly to excite the mode with $\lambda = -2$, we seek a solution in the form $V_C^{zi}(t) = de^{-2t}$; we should have initial conditions $V_C^{zi}(0) = d$ and $\frac{d}{dt}V_C^{zi}(0) = -2d$. Then setting $I_0 = d, V_0 = d$, or $\begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} = d \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, enables us to excite the faster decaying mode.

Response to arbitrary initial conditions

Assume that $\begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$ is given. It is possible to express is any initial condition as a linear combination of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ as shown below:

$$\begin{aligned} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} &= \underbrace{(I_0 - V_0)}_{\alpha} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underbrace{(2V_0 - I_0)}_{\beta} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Hence, every initial condition can be expressed as a linear combination of two special vectors with the combination coefficients α and β. Note that, when either $\alpha = 0$ or $\beta = 0$; we have the single mode excitation. Therefore, it can also be noted from the last equation that we have expressed an arbitrary initial condition as a linear combination of single mode exciting initial conditions. Then the solution for an arbitrary input is as follows:

$$\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

where $\alpha = I_0 - V_0$ and $\beta = 2V_0 - I_0$.

It can be observed if both α and β have the same order of magnitude, then the mode with slower decay rate becomes the dominant mode as $t \to \infty$.

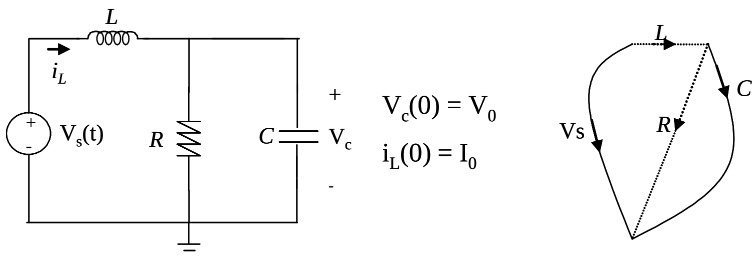

Figure 1.3: An RLC Circuit and its proper tree

1.3.2 Solution by 1st Order Matrix Differential Equations (State Equations)

We examine the same circuit via the state equations. Figure 1.3 shows the circuit and its proper tree.

The state equations of the circuit can be written as follows:

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_s(t) \tag{1.17}$$

The initial conditions for the first order 2x2 matrix differential equation is given as follows:

$$\begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} \tag{1.18}$$

First, we show that two descriptions of the circuit that is the state equation form and node equation form can be retrieved from each other.

To get the node equation form the state equation,

1. Take derivative of the first state equation to get $\ddot{V}_C(t) = \frac{-1}{RC}\dot{V}_C(t) + \frac{1}{C}\dot{I}_L(t)$.
2. Substitute $I_L(t)$ from second state equation to get $\ddot{V}_C(t) = \frac{-1}{RC}\dot{V}_C(t) + \frac{1}{C}(\frac{-1}{L}V_C(t) + \frac{1}{L}V_s(t))$.
3. After moving all terms involving unknowns to the left, we get $\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_C(t) = \frac{V_s(t)}{LC}$.

The step given above show that the two formalisms, i.e. the 2nd order scalar differential equation and the 1st order 2×2 matrix differential equation, are equivalent; that is one can go back and forth between these two descriptions. This does not mean that we should use only one of them. We use whichever is more appropriate for our purposes.

Now let's set $R = 1/3$, $C = 1$ and $L = 1/2$ in the state equations:

$$\begin{bmatrix} \dot{V}_C(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} V_s(t) \tag{1.19}$$

$$\begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} \tag{1.20}$$

Now, we adopt the classical notation for state variables:

$$\mathbf{x(t)} = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} \tag{1.21}$$

The vector $\mathbf{x(t)}$ is called the state vector. The state vector traces the locus of state variables in time. The state equation can be written in the following canonical form

$$\begin{aligned} \dot{\mathbf{x}}(\mathbf{t}) &= \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{b}u(t) \\ \mathbf{x}(\mathbf{0}) &= \mathbf{x}_0 \end{aligned} \tag{1.22}$$

Here $\mathbf{A}$ is a matrix corresponding to $\begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$, $\mathbf{b}$ is a vector corresponding to $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $u(t)$ is the external excitation ($V_s(t)$) and $\mathbf{x}_0$ is the initial condition vector. As we did before, we focus on zero-input response.

The solution of this system is similar to the scalar case. We assume that $\mathbf{x}_{zi}(\mathbf{t}) = \mathbf{c}e^{\lambda t}$ and substitute this guess into the matrix differential equation. Once we do that, we get:

$$\mathbf{c}\lambda e^{\lambda t} = \mathbf{A}\mathbf{c}e^{\lambda t}$$

By dividing both sides by $e^{\lambda t}$ we get,

$$\mathbf{A}\mathbf{c} = \lambda\mathbf{c} \tag{1.23}$$

This shows that the eigenvalues of $\mathbf{A}$ are the natural frequencies that we are looking for.

For the presented example, the eigenvalues of $\mathbf{A}$ are the roots of the following equation:

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \left|\begin{bmatrix} \lambda+3 & -1 \\ 2 & \lambda \end{bmatrix}\right| = \lambda^2+3\lambda+2 \tag{1.24}$$

Hence, they are $\lambda = \{-1,-2\}$. Then the form of the solution is

$$\mathbf{x}_{zi}(\mathbf{t}) = \mathbf{c}e^{-t} + \mathbf{d}e^{-2t} \tag{1.25}$$

Note that if the system examined here is a 2nd order system. If the system an Nth order system the same result is exactly valid. In other words, the natural frequencies are the eigenvalues of $\mathbf{A}$ matrix and the only difference is that $\mathbf{A}$ is a $N \times N$ matrix.

Single Mode Excitation

To excite only one mode, we need to find suitable initial conditions. By setting either $\mathbf{c}$ or $\mathbf{d}$ in $\mathbf{x}_{zi}(\mathbf{t}) = \mathbf{c}e^{-t} + \mathbf{d}e^{-2t}$ to the zero vector, we can find the conditions for single mode excitation.

Let's find the initial conditions to excite the mode with $\lambda = -1$ for the presented example. Then $\mathbf{x}_{zi}(\mathbf{t}) = \mathbf{c}e^{\lambda t}$ and substituting this equation into $\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t})$ (remember that $u(t) = 0$, since this is the zero-input solution), we get

$$\mathbf{A}\mathbf{c} = \lambda\mathbf{c} \tag{1.26}$$

This equation shows that the $\mathbf{c}$ vector is should be the eigenvector of $\mathbf{A}$ for single mode excitation. To find this eigenvector, we solve for $\mathbf{c}$ in $\mathbf{A}\mathbf{c} = \lambda\mathbf{c}$. By moving $\lambda\mathbf{c}$ to the left hand side we get $(\lambda\mathbf{I}-\mathbf{A})\mathbf{c} = \mathbf{0}$. After substituting $\lambda = -1$ into this equation, we get:

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.27}$$

Then we get $\mathbf{c} = \Gamma_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The eigenvector of $\mathbf{A}$ corresponding to the $\lambda = -2$ can be found similarly as $\mathbf{d} = \Gamma_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The variables Γ_1 and Γ_2 are arbitrary scalars, scaling the eigenvectors.

Response to arbitrary initial conditions

Let's assume that initial conditions for the studied problem is given as $\begin{bmatrix} V_C(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$. The given initial conditions pass to $t = 0^+$ as it is, since there are no impulses or switches that are produce a discontinuity in the state variables. The initial conditions at $t = 0^+$ can be written as a linear combination of eigenvectors of $\mathbf{A}$.

$$\begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \Gamma_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \Gamma_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

From the last equation, we can solve for Γ_1 and Γ_2 to get $\Gamma_1 = 4$ and $\Gamma_2 = 6$. Hence the initial condition vector $\mathbf{x_0}$ can be written as $\mathbf{x_0} = \Gamma_1 \mathbf{e_1} + \Gamma_2 \mathbf{e_2}$. (Remember $\mathbf{e_k}$ is the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue of λ_k.) If either Γ_1 or Γ_2 is equal to zero, then a single mode is excited and the response is $\Gamma_k \mathbf{e_k} e^{\lambda_k t}$; when both modes are excited the response becomes $\mathbf{x}(t) = \Gamma_1 \mathbf{e_1} e^{\lambda_1 t} + \Gamma_2 \mathbf{e_2} e^{\lambda_2 t}$, which is in this case

$$\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \underbrace{\Gamma_1}_{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \exp(\underbrace{\lambda_1}_{-1} t) + \underbrace{\Gamma_2}_{6} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(\underbrace{\lambda_2}_{-2} t)$$

Figure 1.4 shows the state trajectories of the system. In other words, this figure shows the locus of state variables as time progresses. The circles on the curve indicates the position of the state variables at the indicated time. It should be clear that all three curves approach the origin as time progresses, but their "speed" to reach the final destination is different from each other. Two of the three curves shown in this figure correspond to single mode excitation. The top and the bottom curves are actually straight lines and the points on these lines can corresponding to the initial conditions exciting a single mode. If initial condition happens to be on these straight lines, the response approaches the origin on these lines, i.e. $I_L(t)/V_C(t) = I_L(0^+)/V_C(0^+)$ for all t. We can also note that the curve corresponding to the mode with $\lambda = -2$ or the response with e^{-2t} term decays faster than the other mode.

The third curve in the figure corresponds to an initial value of $\begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$. Note that this curve is not a straight line. As discussed before, the response to this initial condition can be expressed as superposition of two modes. Hence the response to this initial condition has a fast decaying mode and a slow decaying moving mode. As time progresses, the faster moving mode fades away and we are left with slower moving mode. You can note that the middle curve gets closer to the slower decaying mode as time progresses.

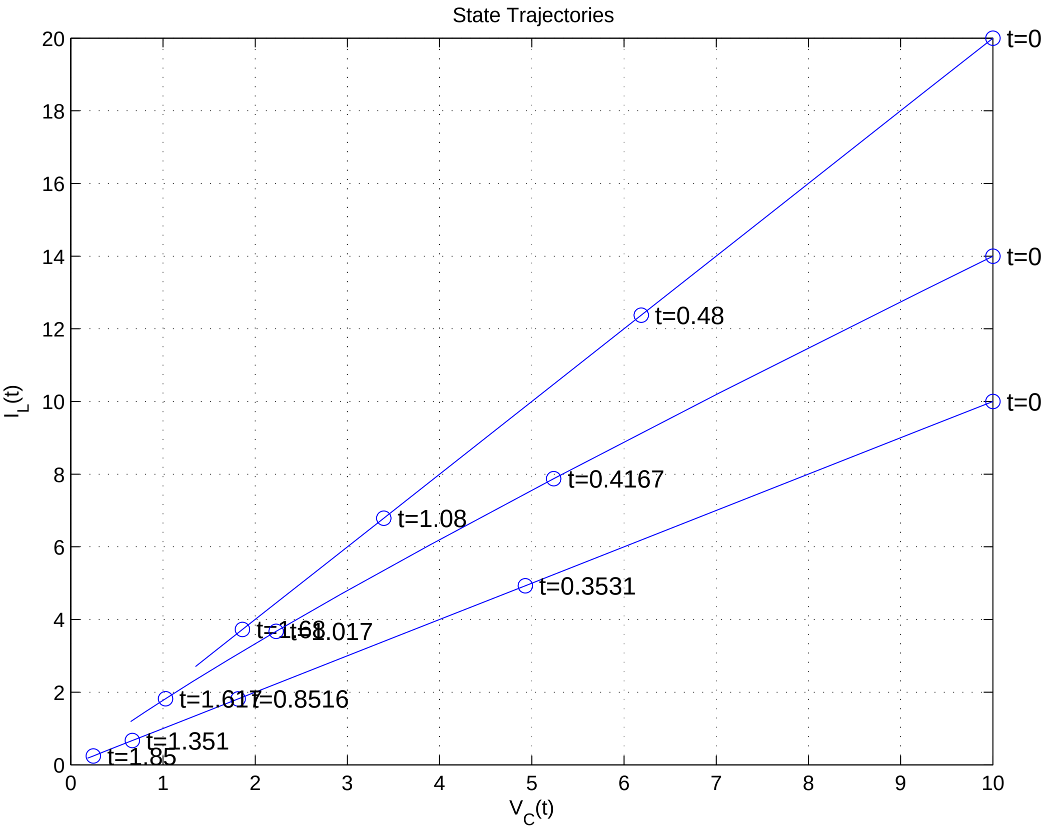

Figure 1.4: State trajectories of the system

1.3.3 Particular Solution

The particular solution of a differential equation is the part of the solution due to the external input. In this section, we examine the the methods of finding particular solution. The main problem presented in this section is finding the response to the exponential input.

The Complex Exponential Function

The complex exponential function is the generalization of conventional exponential function to the complex variables. The conventional exponential function is defined as follows:

$$e^x = \sum_k \frac{x^k}{k!}, \qquad \forall x \in \mathbb{R}$$

The expansion above is called the power series expansion of the exponential function and the region of convergence is the whole real line. The complex exponential is the extension of the same operation to the complex field:

$$e^z = \sum_k \frac{z^k}{k!}, \qquad \forall z \in \mathbb{C}$$

We focus on the exponential function $f(t) = e^{\lambda t}$. Here t is a real variable and λ is a complex number. We view this function as a complex valued function of a real variable. (Interested students can also research complex valued and complex variable functions on the internet. These functions are especially important in electromagnetics and signal processing and some definitions of the specific for the complex calculus is considered as more fundamental than their real valued special cases.)

We examine 5 important special cases of $f(t) = e^{\lambda t}$:

1. λ is real and $\lambda > 0$, i.e. $\lambda = 2$, $f(t) = e^{2t}$. As t increases, the function increases without a bound.

2. λ is real and $\lambda < 0$, i.e. $\lambda = -1$, $f(t) = e^{-t}$. As t increases, the function exponentially decays to zero.

3. λ is purely imaginary, i.e. $z = j\omega, (\omega \in \mathbb{R})$, $f(t) = e^{j\omega t}$. Remembering the Euler's formula ($e^{j\phi} = \cos(\phi) + j\sin(\phi)$); we can express $f(t)$ as

   $f(t) = \cos(\omega t) + j\sin(\omega t)$. This is an example of real variable, complex valued functions.

4. λ is zero, $f(t) = 1$.

5. λ has non-zero real and imaginary parts, i.e. $z = \sigma + j\omega$, $f(t) = e^{\sigma t}e^{j\omega t}$, both σ and ω are assumed to be real. Then $f(t)$ is the multiplication of the cases 1 and 3 (or the multiplication of cases 2 and 3). The magnitude of $f(t)$ which is $|f(t)| = e^{\sigma t}$ which is either a decaying exponential or a blowing up exponential.

The cases of listed above cover a fairly general class of inputs called as the family of exponential inputs. Below, we focus on the response due to the exponential signals. This type of inputs has an intimate connection with the LTI systems which we demonstrate in the next section.

Finding the Particular Response of Nth Order Dynamical Systems for Complex Exponential Input

We continue with the example presented earlier. The circuit shown in Figure 1.1 is described by the following differential equation:

$$\left(D^2 + 3D + 2\right) V_C^p(t) = 2V_s(t) \tag{1.28}$$

We do not state the initial conditions, since our goal is the study of the particular solution. $V_C^p(t)$ is the particular solution for the input $V_s(t)$. We assume that $V_s(t) = e^{st}$ and $s \in \mathbb{C}$.

Our method of solution is the simplest possible one. We make an educated guess on the solution and claim the following:

$$V_C^p(t) = Ae^{st}$$

Here A is the unknown parameter of the solution which can be a complex number in general. Note that the guess is in the same of form of the excitation! (The educated guess is due to well known properties of the exponential function under the differentiation, that is $De^{st} = se^{st}$. Considering e^{st} as the input function at the specific "frequency" of s, then the output of the differentiation system is the input times s. In other words, at the differentiation output the function is only scaled by s, its form remains intact. The functions whose form does not change after a transformation/operator are called the eigenfunctions of that transformation. Hence the exponential function is eigenfunction of the derivative operator. Given all these, the solution of (1.28) when the right hand side of this equation $V_s(t) = e^{st}$ can be $V_C^p(t) = Ae^{st}$, since you can not get rid off an exponential by taking its derivative and summing up by other exponentials!)

To find the unknown A, we substitute the guess $V_C^p(t) = Ae^{st}$ into the differential equation, $(D^2 + 3D + 2)\, V_C^p(t) = 2e^{st}$ and get the following:

$$\left(s^2 + 3s + 2\right) Ae^{st} = 2e^{st}$$

From the last equation, we get the unknown A as

$$A = \frac{2}{s^2 + 3s + 2}$$

Then the solution becomes

$$V_C^p(t) = \frac{2}{s^2 + 3s + 2} e^{st}$$

Remember that the solution for the differential equations is unique under fairly general conditions, therefore we do not need to look any further.

An important note is that when $s = \{-1, -2\}$, that is when the external input is e^{-t} or e^{-2t}, the guess of $V_C^p(t) = Ae^{st}$ does not work. (A becomes undefined due to the zero in the denominator of A.) In other words, when the input excitation matches the natural frequencies, we should do something else, probably make another guess. (As you may remember from the differential equations course, the guess should be corrected to $Ate^{-t} + Be^{-t}$.)

Now we examine some special cases:

Case 1. $V_s(t) = e^{-5t}$

Our goal is finding the (particular) solution of the following equation, $(D^2 + 3D + 2)\, V_C^p(t) = 2e^{-5t}$. This is the special case of the general problem described earlier, but for clarity let's repeat the steps one more time. Assume a solution in the form $V_C^p(t) = Ae^{-5t}$. Here A is the only unknown of the particular solution. Note the following, $\frac{d}{dt} V_C^p(t) = -5Ae^{-5t}$ and $\frac{d^2}{dt^2} V_C^p(t) = 25Ae^{-5t}$. (The repeated application of the differentiation on the exponential function result in the same exponential exponential apart from scaling.) Then substituting the guess function into differential equation, we get $A = \frac{1}{6}$.

The complete solution is then

$$\begin{aligned} v_{\text{complete}}(t) &= v^h(t) + v^p(t) \\ &= c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{6} e^{-5t} \end{aligned}$$

The complete solution can be finalized by setting the c_1 and c_2 to meet the initial conditions. In this section, we are only interested in the particular solution and do not pursue any further steps towards the complete solution.

We would like to remind that the part of the complete solution that "remains" as $t \to \infty$ is called *the steady-state solution*. The part of the solution that decays to zero as t progresses is the transient solution. For this solution, the only possible steady-state solution is the zero function. The whole solution decays to zero.

Case 2. $V_s(t) = 1$

By setting $s = 0$ in $V_s(t) = e^{st}$, we get $V_s(t) = 1$. Hence the important DC input case is also covered by the family of the exponential inputs. The particular solution for the differential equation $(D^2 + 3D + 2)\, V_C^p(t) = 2$ is $V_C^p(t) = \frac{1}{2}$.

This solution can be found by setting $s = 0$ in the general formula of $V_C^p(t) = \frac{2}{s^2+3s+2} e^{st}$. The complete solution is then $v_{\text{complete}}(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}$. It is clear that $f(t) = \frac{1}{2}$ is the steady-state solution for this system. Hence for this case the particular is the steady-state solution.

Before proceeding with other examples, we would like to show a short-cut method for finding the particular solution without expressing the differential equation. Figure 1.5 shows the original circuit with the guess of $V_C^p(t)$ substituted for the capacitor voltage.

We apply the conventional time-domain description of each component to find its branch current and voltage. In the circuit shown above KVL and KCL should

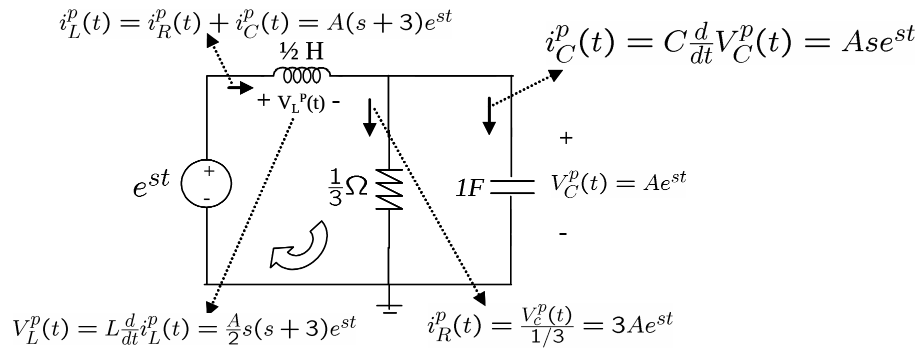

Figure 1.5: Finding the particular solution on the circuit diagram.

be satisfied for the complete response, not for the particular response. But our previous analysis results (cases 1 and 2) imply that the complete solution is in the form $v_{complete}(t) = c_1 e^{-t} + c_2 e^{-2t} + Ae^{st}$ and in the circuit diagram presented in Figure 1.5, we only illustrate the particular components of the solution. We know that there exists a homogeneous solution accompanying each particular solution at every branch.

Since KVL and KCL should be satisfied for $\forall t$, it should be clear that the terms involving only e^{st} should satisfy KVL and KCL equations on their own. (This is due to independence of functions e^{st}, e^{-t} and e^{-2t}.) Because of this reason, we only indicate the particular solution on the circuit diagram shown in Figure 1.5. Finally writing a KVL equation for the loop of voltage source, inductor and capacitor, we get an equation as follows:

$$\begin{aligned} e^{st} &= V_L^p(t) + V_L^p(t) \\ &= \frac{A}{2}s(s+3)e^{st} + Ae^{st} \\ &= \frac{A}{2}(s^2+3s+2)e^{st} \end{aligned}$$

From the last equation, we get $A = \frac{2}{s^2+3s+2}$ and find the particular solution as before $V_C^p(t) = \frac{2}{s^2+3s+2}e^{st}$. This method demonstrates the ease of approach to find the particular solution for exponential input family. We do not even need to find the differential equation to find the particular solution!

Case 3. $V_s(t) = \cos(5t)$

We make the following guess for the particular solution $V_C^p(t) = A\cos(5t) + B\sin(5t)$. When we substitute the particular solution into the differential equation of $(D^2 + 3D + 2)\,V_C^p(t) = 2\cos(5t)$, we get:

$$(-25A + 15B + 2A)\cos(5t) + (-25B - 15A + 2B)\sin(5t) = 2\cos(5t)$$

To satisfy this equation for $\forall t$, we need to equate the coefficients of the independent functions (sine and cosine) at right and left side of the equation. Once we do that, we get two equations for two unknowns of A and B.

$$\begin{bmatrix} -23 & 15 \\ -15 & -23 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

From the last equation, we can find $A = \frac{-46}{23^2+15^2}$, $B = \frac{30}{23^2+15^2}$ and the particular solution is then $V_C^p(t) = \frac{-46}{23^2+15^2}\cos(5t) + \frac{30}{23^2+15^2}\sin(5t)$.

Now, we present a faster procedure. We choose to write $V_s(t) = \cos(5t)$ as $V_s(t) = \mathrm{Re}\{e^{j5t}\}$. Then the guess for the particular solution is the following $V_C^p(t) = \mathrm{Re}\{Ae^{j5t}\}$. We should substitute this guess into $(D^2 + 3D + 2)\, V_C^p(t) = 2\mathrm{Re}\{e^{j5t}\}$ and find A. (Note that A for this guess is a complex number different from the earlier cases.)

We would like to mention the following simple fact on the derivative of the complex valued functions. Considering the application of $fracddt$ to e^{j5t}. The derivative operators on the real and imaginary parts of the argument, that is $\frac{d}{dt}e^{j5t} = \frac{d}{dt}\cos(5t) + j\frac{d}{dt}\sin(5t)$. Then the following relation $\frac{d}{dt}\mathrm{Re}\{e^{j5t}\} = \mathrm{Re}\{\frac{d}{dt}e^{j5t}\}$ is true.

Then substituting $V_C^p(t) = \mathrm{Re}\{Ae^{j5t}\}$ for $(D^2 + 3D + 2)\, V_C^p(t) = 2\mathrm{Re}\{e^{j5t}\}$ we get the following:

$$\begin{aligned}
\frac{d^2}{dt^2}\mathrm{Re}\{Ae^{j5t}\} + 3\frac{d}{dt}\mathrm{Re}\{Ae^{j5t}\} + 2\mathrm{Re}\{Ae^{j5t}\} &= 2\mathrm{Re}\{e^{j5t}\} \\
\mathrm{Re}\{A\frac{d^2}{dt^2}e^{j5t}\} + \mathrm{Re}\{3A\frac{d}{dt}e^{j5t}\} + \mathrm{Re}\{2Ae^{j5t}\} &= \mathrm{Re}\{2e^{j5t}\} \\
\mathrm{Re}\{(j5)^2A\frac{d^2}{dt^2}e^{j5t}\} + \mathrm{Re}\{3(j5)A\frac{d}{dt}e^{j5t}\} + \mathrm{Re}\{2Ae^{j5t}\} &= \mathrm{Re}\{2e^{j5t}\} \\
\mathrm{Re}\{(j5)^2Ae^{j5t} + 3(j5)3Ae^{j5t} + 2Ae^{j5t}\} &= \mathrm{Re}\{2e^{j5t}\} \\
\mathrm{Re}\{e^{j5t}A((j5)^2 + 3(j5) + 2)\} &= \mathrm{Re}\{2e^{j5t}\}
\end{aligned}$$

The last equation is obviously satisfied with the choice of

$$\begin{aligned}
A &= \frac{2}{(j5)^2 + 3(j5) + 2} \\
&= \frac{2}{-23 + j15} \\
&= \frac{2(-23 - j15)}{23^2 + 15^2}
\end{aligned}$$

Then the solution becomes

$$\begin{aligned} V_C^p(t) &= \mathrm{Re}\{Ae^{j5t}\} \\ &= \mathrm{Re}\left\{\frac{2(-23-j15)}{23^2+15^2}e^{j5t}\right\} \\ &= \frac{2}{23^2+15^2}\mathrm{Re}\left\{(-23-j15)(\cos 5t + j\sin 5t)\right\} \\ &= \frac{2}{23^2+15^2}\left(-23\cos 5t + 15\sin 5t\right) \end{aligned}$$

which is exactly the same equation that we have got before. The method with the complex number is the basis of AC circuit analysis which is to be examined later.

1.4 Stability

The concept of stability is fundamental for the study of the dynamical systems. In circuit theory, we introduce the concepts of asymptotically stable, stable and unstable circuits. These concepts are build on the behaviour of the circuit due to initial conditions. After the study of these types of stability, we also introduce the BIBO (Bounded Input Bounded Output) stability which is fundamental for the study linear systems with input.

We illustrate the stability concept with the following 2nd order example:

$$\left(D^2 - (\lambda_1+\lambda_2)D + \lambda_1\lambda_2\right)x(t) = f(t)$$

Here we have two natural frequencies at $\lambda = \{\lambda_1, \lambda_2\}$ which can be complex numbers. The function $f(t)$ is the forcing term of the differential equation.

Asymptotical Stability

A system is called *asymptotically stable* if the response due to any initial condition decays to zero as $t \to \infty$. For example, $x^h(t) = c_1e^{-t} + c_2e^{-2t}$ goes to zero for any initial condition. Therefore a system with such a homogeneous solution or natural frequencies of -1 and -2 is an example of asymptotically stable systems. The condition for the asymptotical stability can be expressed as follows:

$$\mathrm{Re}\{\lambda_k\} < 0\ ,\quad \forall\lambda_k \ \Leftrightarrow\ \text{Asymptotically Stable}$$

Stability

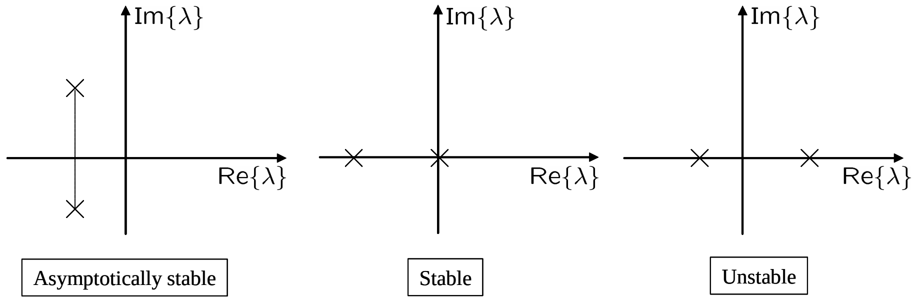

Figure 1.6: Definitions of stability. The crosses show the position of natural frequencies on the complex plane.

A system is called *stable* if the response to initial conditions remains bounded as $t \to \infty$. (A function is called bounded if $|f(t)| < M$ for a finite M.) For example, $x^h(t) = c_1 + c_2 e^{-2t}$ is a stable system. In general, the condition can be expressed as follows:

$$\text{Re}\{\lambda_k\} \leq 0 \ , \quad \forall \lambda_k \ \Leftrightarrow \ \text{Stable}$$

Note that if the system has a purely imaginary natural frequency or have a natural frequency of $\lambda = 0$; then the system is said to be stable but not asymptotically stable.

Unstable

A system is called *unstable* if the response to the initial condition is unbounded as $t \to \infty$. An example for an unbounded function can be $x^h(t) = c_1 + c_2 e^{2t}$. The general condition can be expressed as follows:

$$\text{Re}\{\lambda_k\} > 0 \ , \quad \exists \lambda_k \ \Leftrightarrow \ \text{Unstable}$$

Figure 1.6 illustrates the definitions of stability for a set of natural frequencies on the complex plane.

BIBO Stability

A system with input $f(t)$and output $g(t)$ is stable if every bounded $f(t)$ results in a bounded $g(t)$. As the name implies the bounded input results in a bounded output. Considering the exponential inputs, the complete solution due to exponential inputs is as follows:

$$g_{\text{complete}}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + A e^{st}$$

Here e^{st} is the input function and λ_1 and λ_2 are the natural frequencies as before. We assume for now that $s \neq \{\lambda_1, \lambda_2\}$. Input being bounded means $s \leq 0$, then

1. If either $Re\{\lambda_1\}$ or $Re\{\lambda_2\}$ is positive (an unstable system as shown in Figure 1.6), then the output is unbounded. Therefore the system is *not* BIBO stable.

2. If both $Re\{\lambda_1\}$ and $Re\{\lambda_2\}$ is negative (an asymptotically stable system as shown in Figure 1.6), then the output is bounded. Therefore the system is BIBO stable.

3. If either $Re\{\lambda_1\}$ or $Re\{\lambda_2\}$ (or both) is equal to zero (a stable system as shown in Figure 1.6), then the output is unbounded either for $s = \lambda_1$ or $s = \lambda_2$. As an example assume that $\lambda_1 = -5$ and $\lambda_2 = 0$, and the input $f(t) = 1$ then the output becomes

$$g_{complete}(t) = c_1 e^{-5t} + c_2 + A + Bt$$

   The response is unbounded. The system is not BIBO stable.