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Vibrations and Waves

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THE M.I.T. INTRODUCTORY PHYSICS SERIES

4 Forced vibrations and resonance

THE PRECEDING CHAPTER was concerned entirely with the free vibrations of various types of physical systems. We shall now turn to the remarkable phenomena, of profound importance throughout physics, that occur when such a system—a physical oscillator—is subjected to a periodic driving force by an external agency.

The key word is "resonance." Everybody has at least a qualitative familiarity with this phenomenon, and probably the most striking feature of a driven oscillator is the way in which a periodic force of a fixed size produces very different results depending on its frequency. In particular, if the driving frequency is made close to the natural frequency, then (as anyone who has pushed a swing knows) the amplitude of oscillation can be made very large by repeated applications of a quite small force. This is the phenomenon of resonance. A force of about the same size at frequencies well above or well below the resonant frequency is much less effective; the amplitude produced by it remains quite small. To judge by the quotation at the beginning of this chapter, the phenomenon has been recognized for a very long time.[1] It

[1]As Alexander Wood remarks in his book *Acoustics* (Blackie & Son, London, 1940): "It seems difficult to believe that legislation should be designed to cover a situation that had never arisen." The example does seem rather bizarre, however, and H. Bouasse, the French physicist who drew attention to this Talmudic pronouncement, reported that he had himself reared a large number of cocks, none of which developed a habit of putting their heads inside glass vases!

is typical of this type of motion that the driven system is compelled to accept whatever repetition frequency the driving force has; *its tendency to vibrate at its own natural frequency may be* in evidence at first, but ultimately gives way to the external influence.

To provide some initial feeling for the theoretical description of the resonance phenomenon, without getting too involved with analytical details, we shall begin by considering the simple though physically unreal case of an oscillator in which the damping effect is entirely negligible.

UNDAMPED OSCILLATOR WITH HARMONIC FORCING

We shall take our system to be the usual mass m on a spring of spring constant k. To this we shall imagine the application of a sinusoidal driving force $F = F_0 \cos \omega t$. The value of $\sqrt{k/m}$, representing the natural angular frequency of the system, will be denoted by ω_0. Then the statement of the equation of motion, in the form $ma =$ net force, is

$$m \frac{d^2x}{dt^2} = -kx + F_0 \cos \omega t$$

or

$$m \frac{d^2x}{dt^2} + kx = F_0 \cos \omega t \tag{4-1}$$

Before we discuss this differential equation of motion in detail, let us consider the situation qualitatively. If the oscillator is driven from its equilibrium position and then left to itself, it will oscillate with its natural frequency ω_0. A periodic driving force will, however, try to impose its own frequency[1] ω on the oscillator. We must expect, therefore, that the actual motion in this case is some kind of a superposition of oscillations at the two frequencies ω and ω_0. The mathematically complete solution of Eq. (4–1) is indeed a simple sum of these two motions. But because of the inevitable presence of dissipative forces in any real system, the free oscillations will eventually die out. The initial stage, in which the two types of motion are both prominent, is called the *transient*. After a sufficiently long time, however, the only motion in effect present is the forced oscillation, which will continue undiminished at the frequency ω. When this condition has been achieved, we

[1]To avoid tiresome repetitions, we shall often refer to ω simply as "frequency" rather than "angular frequency" in contexts where no ambiguity is entailed.

have what is called a steady-state motion of the driven oscillator.

Later we shall analyze the transient effects, but for the present we shall focus our attention exclusively on the steady state of the forced oscillation. In an ideal undamped oscillator, the effect of the natural vibrations would never disappear, but we shall temporarily ignore this embarrassing fact for the sake of the simplicity that absence of damping brings to the forced-motion problem.

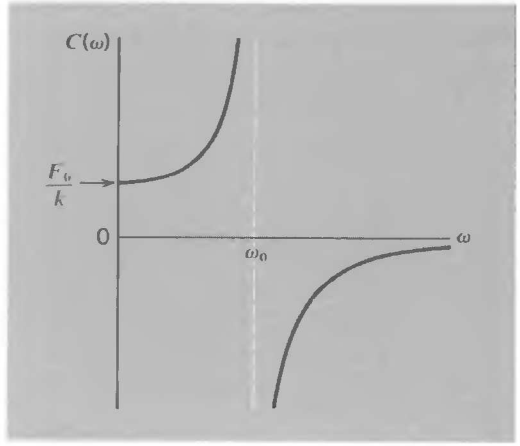
The most striking feature of the motion will be the large response near $\omega = \omega_0$, but before embarking on the solution of Eq. (4–1) in its entirety, let us point to some features of the motion in the extremes of very low or very high values of the driving frequency ω. If the driving force is of very low frequency relative to the natural frequency of free oscillations, we would expect the particle to move essentially in step with the driving force with an amplitude not very different from F_0/k $(= F_0/m\omega_0^2)$, the displacement which a constant force F_0 would produce. This is equivalent to stating that the term $m(d^2x/dt^2)$ in Eq. (4–1) plays a relatively small role compared to the term kx at very low frequencies, or in other words that the response is controlled by the stiffness of the spring. On the other hand, at frequencies of the driving force very large compared to the natural frequency of free oscillation, the opposite situation holds. The term kx becomes small compared to $m(d^2x/dt^2)$ because of the large acceleration associated with high frequencies, so that the response is controlled by the inertia. In this case we expect a relatively small amplitude of oscillation and this oscillation should be opposite in phase to the driving force, because the acceleration of a particle in harmonic motion is 180° out of phase with its displacement. It is still not apparent from these remarks that the resonant amplitude should greatly exceed that at low or high frequencies, but this we shall now show.

To obtain the steady-state solution of Eq. (4–1) we set

$$x = C \cos \omega t \tag{4–2}$$

We are assuming, in other words, that the motion is harmonic, of the same frequency and phase as the driving force, and that the natural oscillations of the system are not present. It must be kept in mind that the assumption of Eq. (4–2) is tentative and we must be prepared to reject it if we fail to find a value of the as-yet-undetermined constant C such that Eq. (4–1) is satisfied for arbitrary values of ω and t. Differentiating Eq. (4–2) twice

Fig. 4–1 Amplitude of forced oscillations as a function of the driving frequency (assuming zero damping.) The negative sign of the amplitude for $\omega > \omega_0$ corresponds to a phase lag π of displacement with respect to driving force.



with respect to t, we get

$$\frac{d^2x}{dt^2} = -\omega^2 C \cos \omega t$$

Substituting in Eq. (4–1) we thus have

$$-m\omega^2 C \cos \omega t + kC \cos \omega t = F_0 \cos \omega t$$

and hence

$$C = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2} \tag{4–3}$$

Equation (4–3) satisfactorily defines C in such a way that Eq. (4–1) is always satisfied. Thus we can take it that the forced motion is indeed described by Eq. (4–2), with C depending on ω according to Eq. (4–3). This dependence is shown graphically in Fig. 4–1. Notice how C switches abruptly from large positive to large negative values as ω passes through ω_0. The resonance phenomenon itself is represented by the result that the magnitude of C, without regard to sign, becomes infinitely large at $\omega = \omega_0$ exactly.

Although Eqs. (4–2) and (4–3) between them describe in a perfectly adequate way the solution of this dynamical problem, there is a better way of stating the result, more in accord with our general description of harmonic motions. This is to express x in terms of a sinusoidal vibration having an amplitude A, by definition a *positive* quantity, and a phase α at $t = 0$.

$$x = A \cos(\omega t + \alpha) \tag{4–4}$$

It is not difficult to see that this implies putting $A = |C|$ and giving α one or other of two values, according to whether the driving frequency ω is less or greater than ω_0:

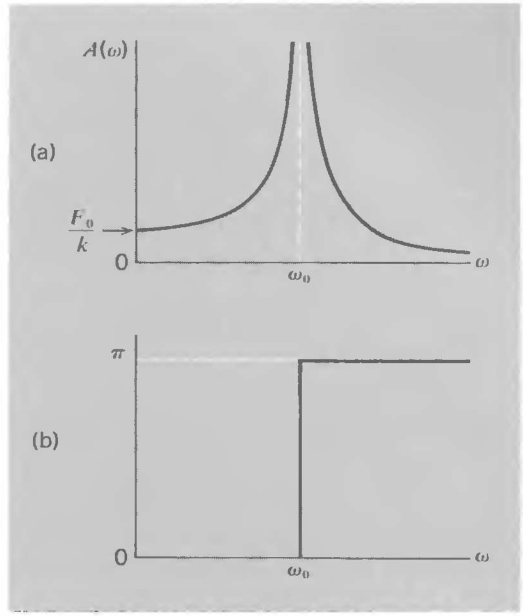


Fig 4–2 (a) Absolute amplitude of forced oscillations as a function of the driving frequency, for zero damping.
(b) Phase lag of the displacement with respect to the driving force as a function of frequency.

$$\omega < \omega_0: \alpha = 0$$
$$\omega > \omega_0: \alpha = \pi$$

The response of the system over the whole range of ω is then represented by separate curves for the amplitude A and the phase α, as shown in Fig. 4–2. The infinite value of A at $\omega = \omega_0$, and the discontinuous jump from zero to π in the value of α as one passes through ω_0, must be unphysical, but, as we shall see, they represent a mathematically limiting case of what actually occurs in systems with nonzero damping.

The actual reversal of phase of the displacement with respect to the driving force (i.e., from being in phase to being 180° out

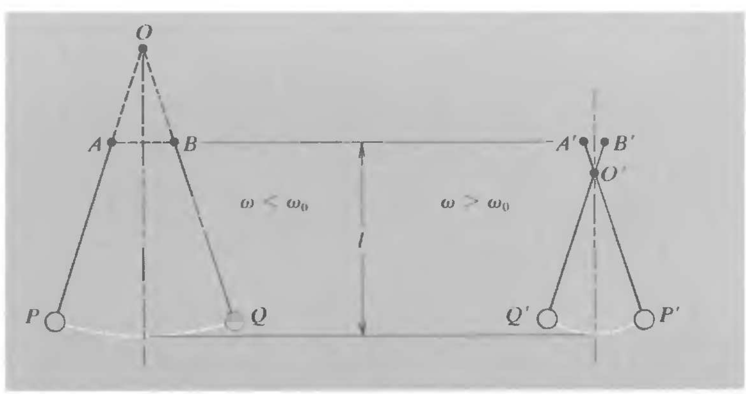


Fig. 4–3 Motion of simple pendulums resulting from forced harmonic oscillation of the point of suspension along the line AB. (a) $\omega < \omega_0$. (b) $\omega > \omega_0$.

of phase) is shown in a very direct way by the behavior of a simple pendulum that is driven by moving its point of suspension back and forth horizontally in SHM. The situations for frequencies well below and well above resonance are illustrated in Fig. 4–3. Once the steady state has been established, the pendulum behaves as though it were suspended from a fixed point corresponding to a length greater than its true length l for $\omega < \omega_0$, and less than l for $\omega > \omega_0$. In the former case the motion of the bob is always in the same direction as the motion of the suspension, whereas in the latter case it is always opposite.

THE COMPLEX EXPONENTIAL METHOD FOR FORCED OSCILLATIONS

Having dealt with this simplest of forced vibration problems in terms of sinusoidal functions, let us do it again using the complex exponential. This has no special merit as far as the present problem is concerned, but the technique, illustrated here in elementary terms, will show to great advantage when we come to deal with the damped oscillator. Our program is as follows:

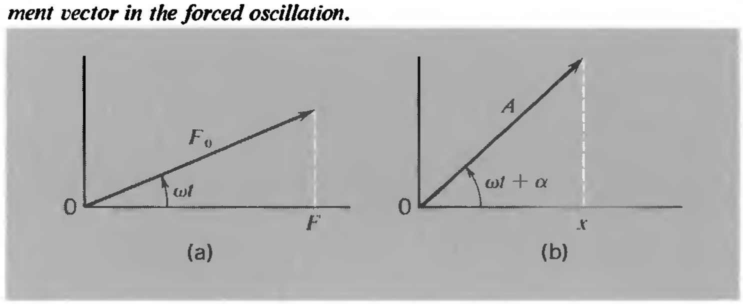
1. We start with the physical equation of motion as given by Eq. (4–1):

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega t$$

2. We imagine the driving force $F_0 \cos \omega t$ as being the projection on the x axis of a rotating vector $F_0 \exp(j\omega t)$, as shown in Fig. 4–4(a), and we imagine x as being the projection of a vector z that rotates at the same frequency ω [Fig. 4–4(b)].

3. We then write the differential equation that governs z:

Fig. 4–4 (a) Complex representation of sinusoidal driving force. (b) Complex representation of displacement vector in the forced oscillation.



$$m\frac{d^2z}{dt^2} + kz = F_0e^{j\omega t} \tag{4-5}$$

4. We try the solution

$$z = Ae^{j(\omega t+\alpha)}$$

Substituting in Eq. (4–5) this gives us

$$(-m\omega^2 A + kA)e^{j(\omega t+\alpha)} = F_0e^{j\omega t}$$

which can be rewritten as follows:

$$(\omega_0{}^2 - \omega^2)A = \frac{F_0}{m}e^{-j\alpha}$$

$$= \frac{F_0}{m}\cos\alpha - j\frac{F_0}{m}\sin\alpha \tag{4-6}$$

This contains two conditions, corresponding to the real and imaginary parts on the two sides of the equation:

$$(\omega_0{}^2 - \omega^2)A = \frac{F_0}{m}\cos\alpha$$

$$0 = -\frac{F_0}{m}\sin\alpha$$

These clearly lead at once to the solutions represented by the two graphs in Fig. 4–2.

FORCED OSCILLATIONS WITH DAMPING

At the end of Chapter 3 we analyzed the free vibrations of a mass–spring system subject to a resistive force proportional to velocity. We shall now consider the result of acting on such a system with a force just like that considered in the previous section. The statement of Newton's law then becomes

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0\cos\omega t$$

or

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t$$

Putting $k/m = \omega_0{}^2$, $b/m = \gamma$, this can be written

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2x = \frac{F_0}{m}\cos\omega t \tag{4-7}$$

Let us now look for a steady-state solution to this equation.

We shall go at once to the complex-exponential method; our basic equation then becomes the following:

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m}e^{j\omega t} \tag{4-8}$$

We shall now assume the following solution:

$$z = Ae^{j(\omega t - \delta)} \tag{4-9}$$

with

$$x = \text{Re}\,(z)$$

Notice that we have assumed a slightly different equation for z than we did in the previous section; we have written the initial phase of z as $-\delta$ instead of $+\alpha$. Why did we do this? The clue is to be found in Eq. (4–6). The right-hand side of the equation can be read, in geometrical terms, as an instruction to take a vector of length F_0/m and rotate it through the angle $-\alpha$ with respect to the real axis. We are going to get a very similar equation now, and it will simplify things if we define our angle, formally at least, as representing a positive (counterclockwise) rotation. That is, δ is formally a positive phase angle by which the driving force leads the displacement.

Substituting from Eq. (4–9) into Eq. (4–8) we thus get

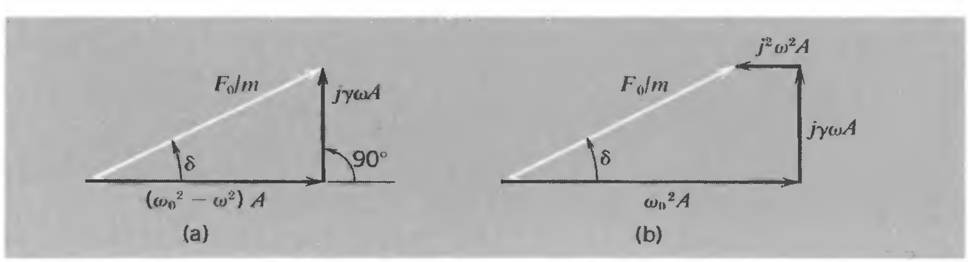
$$(-\omega^2 A + j\gamma\omega A + \omega_0^2 A)e^{j(\omega t - \delta)} = \frac{F_0}{m}e^{j\omega t}$$

Therefore,

$$(\omega_0^2 - \omega^2)A + j\gamma\omega A = \frac{F_0}{m}e^{j\delta} \tag{4-10}$$

Now the elegance and perspicuity of the complex exponential method are really displayed. We can read Eq. (4–10) as a geometrical statement. The left-hand side tells us to draw a vector of length $(\omega_0{}^2 - \omega^2)A$, and then at right angles to it a vector of

Fig. 4–5 Geometrical representation of Eq. (4–10).



length $\gamma\omega A$. The right-hand side tells us to draw a vector of length F_0/m at an angle δ to the real axis. The equation requires that these two operations bring us to the same point, so that the vectors form a closed triangle, as shown in Fig. 4–5(a).[1] Clearly, we have

$$(\omega_0^2 - \omega^2)A = \frac{F_0}{m}\cos\delta$$

$$\gamma\omega A = \frac{F_0}{m}\sin\delta$$

Therefore,

$$A(\omega) = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2}}$$

$$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \qquad (4\text{–}11)$$

These same results can of course be obtained without introducing complex exponentials. One simply assumes a solution of the form

$$x = A\cos(\omega t - \delta) \qquad (4\text{–}12)$$

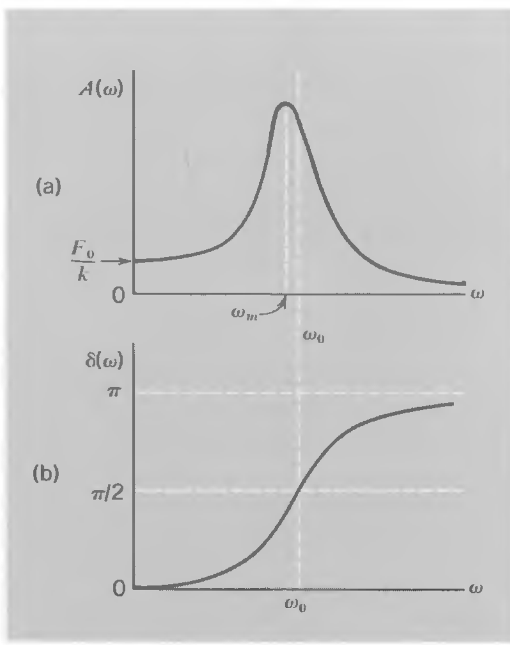


Fig. 4–6 (a) Dependence of amplitude upon driving frequency for forced oscillations with damping. (b) Phase of displacement with respect to driving force as a function of the driving frequency.

[1]You may actually prefer to read the left-hand side of Eq. (4–10) even more literally (in terms of its origins) as a sum of *three* vectors,

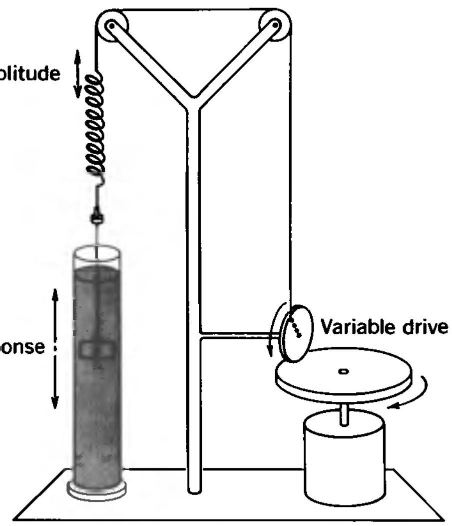
$$\omega_0^2 A + j\gamma\omega A + (j)^2\omega^2 A$$

as shown in Fig. 4–5(b).

and substitutes this in Eq. (4–7), which leads to the equation

$$(\omega_0^2 - \omega^2)A\cos(\omega t - \delta) - \gamma\omega A\sin(\omega t - \delta) = \frac{F_0}{m}\cos\omega t$$

This must then be solved as a trigonometric identity true for all t. The analysis is certainly not difficult, but it is less transparent



(a)

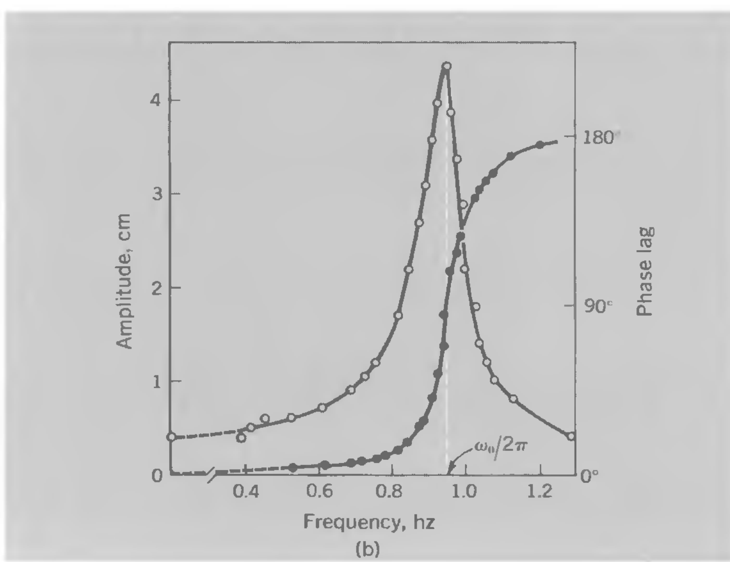


Fig. 4–7 (a) Diagrammatic sketch of the "Texas Tower," a mechanical resonance apparatus developed by J. G. King at the Education Research Center, M.I.T. (b) Experimental resonance curves for amplitude and phase lag obtained with this apparatus. (Measurements by G. J. Churinoff, M.I.T. class of 1967.)

and instructive than the other.

The type of dependence of amplitude A and phase angle δ upon frequency ω, for an assumed constant magnitude of F_0, is shown in Fig. 4–6. (Remember that δ is the angle by which the driving force leads the displacement, or by which the displacement lags behind the driving force.) These curves have a clear general resemblance to those in Fig. 4–2 for the undamped oscillator. As can be seen from the expression for tan δ in equations (4–11), the phase lag increases continuously from zero (at $\omega = 0$) to 180° (in the limit $\omega \rightarrow \infty$); it passes through 90° at *precisely* the frequency ω_0. Less obvious is the fact that the maximum amplitude is attained at a frequency ω_m somewhat less than ω_0; in most cases of any practical interest, however, the difference between ω_m and ω_0 is negligibly small.

These are some of the calculated features of a forced, damped oscillator. How nearly are they exhibited by actual physical systems? Figure 4–7 provides an answer in the form of experimental results obtained with the type of physical system we have been discussing. It is, to be sure, not a natural system but an artificial one, devised specifically to display these features. Nevertheless, there is satisfaction in seeing that the pattern of behavior described by our mathematical analysis (which might, after all, bear no relation to reality) does, in fact, correspond quite well to the behavior of a system containing a real spring and a real viscous damping agency. This is the same system for which we showed the decay of free oscillations in Fig. 3–12.

The features of Fig. 4–6 can also be nicely demonstrated in a simple but, as it were, backhanded way, by applying a driving force of some *fixed* frequency to a whole collection of oscillators of different natural frequencies. This is readily done by a modification of an arrangement due to E. H. Barton (1918) in which a number of light pendulums of different lengths are hung from a horizontal bar that is rocked at the resonance frequency of one pendulum in the middle of the range, as shown in Fig. 4–8(a). When photographed edgewise the motions of the light pendulum bobs, all driven at the same frequency, display, qualitatively at least, the expected phase relationships. This is indicated in Fig. 4–8(b), which shows the displacements of the small pendulums at the instant when the driving bar is passing from left to right through its equilibrium position, and then at a slightly later instant. The short pendulums (for which $\omega_0 > \omega$) have

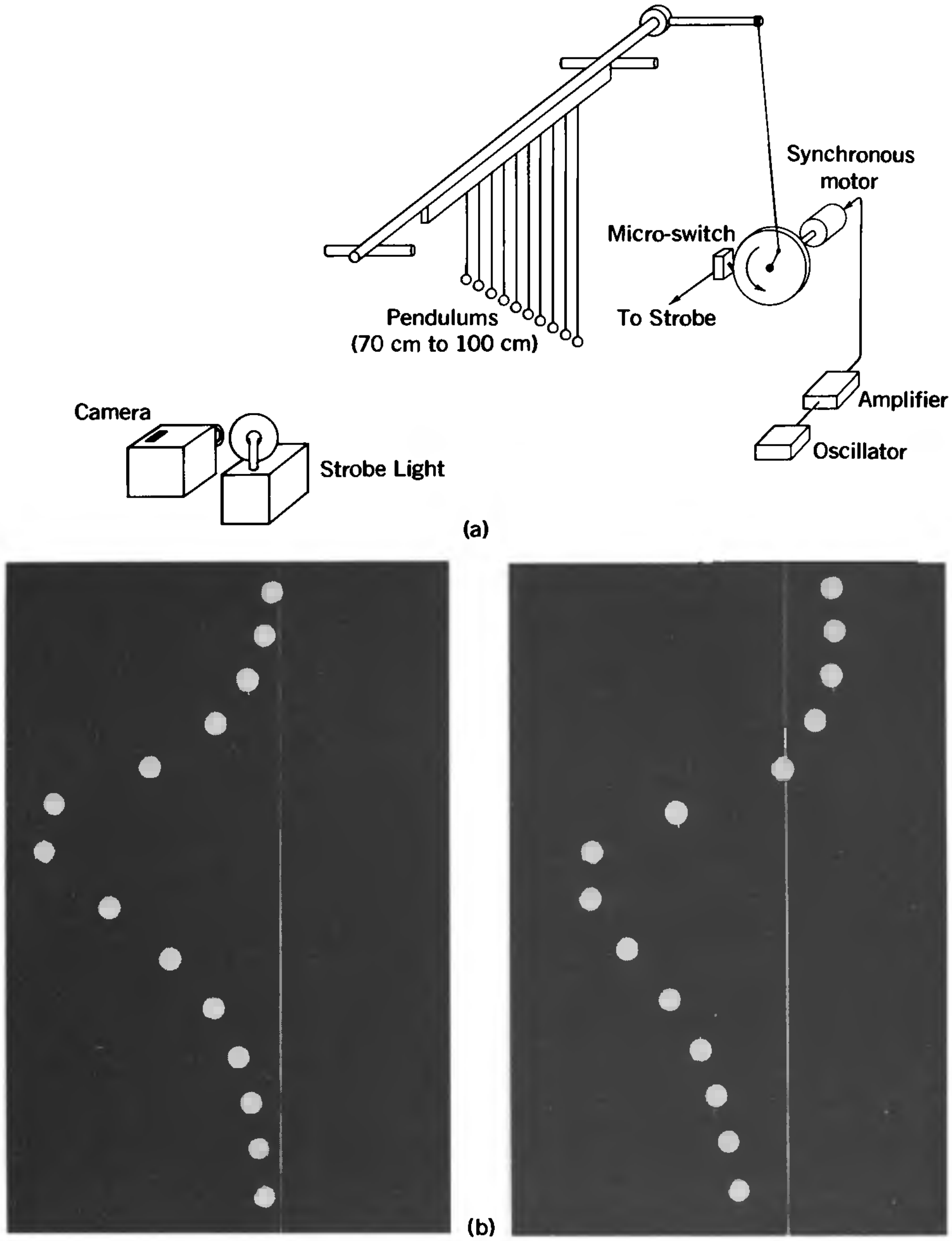


Fig. 4–8 A modern version of Barton's pendulums experiment. (a) A general sketch of the arrangement. The strobe light flashes once per oscillation at a controllable point in the cycle. (b) Displacements of the pendulums when the driving force is passing through zero (left) and at a somewhat later instant (right). In the latter photograph, note that the shorter pendulums have moved in the same direction as the driver and the longer pendulums have moved in the opposite direction, corresponding to $\delta < 90°$ and $\delta > 90°$ respectively. (Photos by Jon Rosenfeld, Education Research Center, M.I.T.).

$\delta < 90°$, the long ones (for which $\omega_0 < \omega$) have $\delta > 90°$, and so move contrary to the driver, and the pendulum in exact resonance lags by 90°, being at maximum negative displacement as the driver passes through zero.

EFFECT OF VARYING THE RESISTIVE TERM

In discussing the decay of free vibrations at the end of Chapter 3, we introduced the "quality factor" Q, the pure number equal to the ratio ω_0/γ. The larger the value of Q, the less the dissipative effect and the greater the number of cycles of free oscillation for a given decrease of amplitude. We shall now indicate how the behavior of the resonant system changes as the Q of the system is changed, other things being equal.

We shall put Eq. (4–11) (for A and $\tan\delta$) into more convenient form for this purpose. First, substituting $\gamma = \omega_0/Q$ gives us

$$\left.\begin{aligned} A(\omega) &= \frac{F_0/m}{[(\omega_0{}^2 - \omega^2)^2 + (\omega\omega_0/Q)^2]^{1/2}} \\ \tan\delta(\omega) &= \frac{\omega\omega_0/Q}{\omega_0{}^2 - \omega^2} \end{aligned}\right. \qquad (4\text{–}13)$$

Furthermore, it will prove convenient for many purposes to use the ratio ω/ω_0, rather than ω itself, as a variable. With this in mind we shall rewrite equations (4–13) in the following form:

$$A = \frac{F_0}{m\omega_0{}^2}\frac{\omega_0/\omega}{\left[\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}\right]^{1/2}}$$

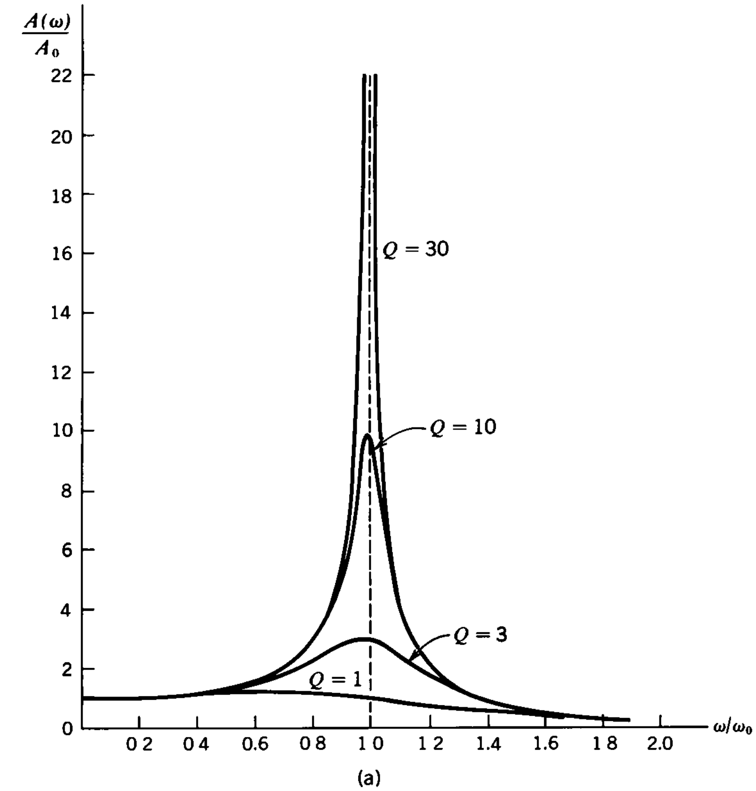
or

$$A = \frac{F_0}{k}\frac{\omega_0/\omega}{\left[\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}\right]^{1/2}} \qquad (4\text{–}14)$$

and

$$\tan\delta = \frac{1/Q}{\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}}$$

In Fig. 4–9 we show curves calculated from equations (4–14) to show the variations with frequency of amplitude A and phase lag δ for different values of Q. Most of the change of δ takes place over a range of frequencies roughly from $\omega_0(1 - 1/Q)$ to $\omega_0(1 + 1/Q)$, i.e., a band of width $2\omega_0/Q$ centered on ω_0. In the limit $Q \to \infty$ the phase lag jumps abruptly from zero to π as



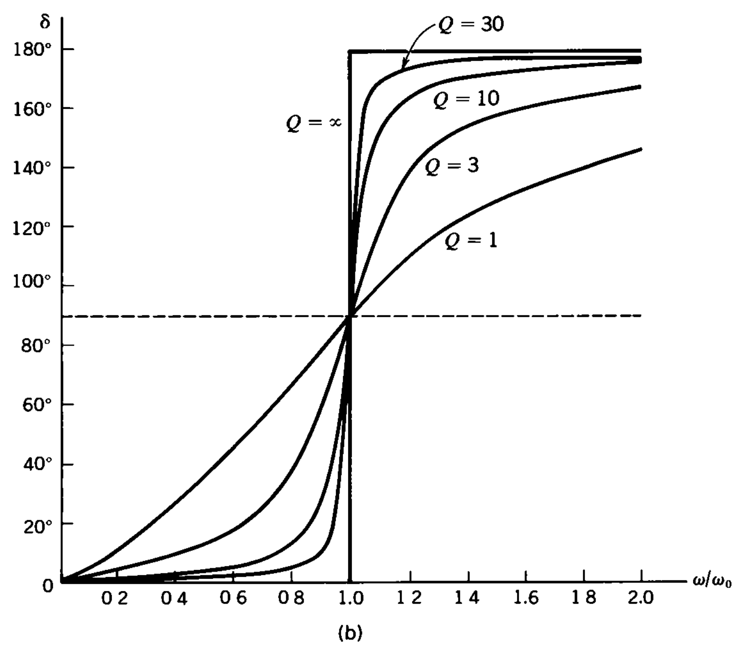


Fig. 4–9 (a) Amplitude as function of driving frequency for different values of Q, assuming driving force of constant magnitude but variable frequency. (b) Phase difference δ as function of driving frequency for different values of Q.

one passes through ω_0. Clearly the frequency ω_0 is an important property of the resonant system, even though it is not (except for zero damping) the frequency with which the system would oscillate when left to itself.

The amplitude A passes through a maximum for any value of Q greater than $1/\sqrt{2}$—i.e., for all except the most heavily damped systems. This maximum amplitude A_m occurs, as we noted earlier, at a frequency ω_m that is less than ω_0. If we denote by A_0 the amplitude F_0/k obtained for $\omega \rightarrow 0$, then one can readily show that the following results hold:

$$\begin{aligned} \omega_m &= \omega_0\left(1 - \frac{1}{2Q^2}\right)^{1/2} \\ A_m &= A_0 \frac{Q}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}} \end{aligned} \tag{4–15}$$

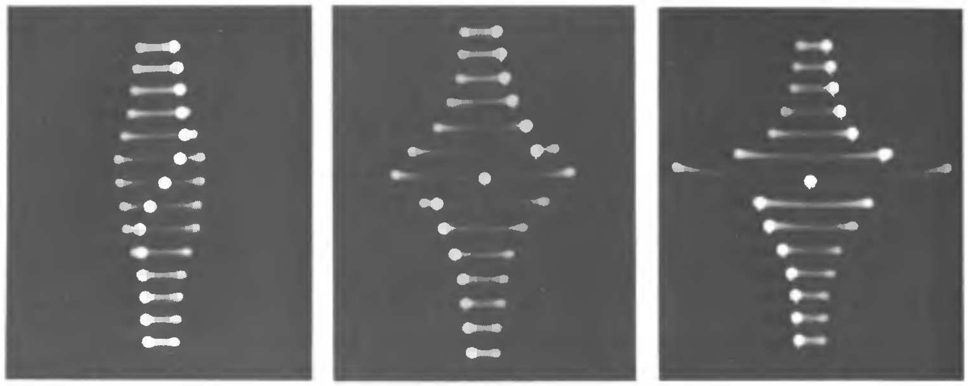
In Table 4–1 we list some values of ω_m/ω_0 and A_m/A_0 for particular Q values. Notice that in most cases ($Q \geq 5$) the peak

TABLE 4–1: RESONANCE PARAMETERS OF DAMPED SYSTEMS

Q	ω_m/ω_0	A_m/A_0
$1/\sqrt{2}$	0	1
1	$1/\sqrt{2} = 0.707$	$2/\sqrt{3} = 1.15$
2	$\sqrt{\frac{7}{8}} = 0.935$	$8/\sqrt{14} = 2.06$
3	$\sqrt{\frac{17}{18}} = 0.973$	$18/\sqrt{35} = 3.04$
5	$\sqrt{\frac{49}{50}} = 0.990$	$50/\sqrt{99} = 5.03$
$\gg 1$	$1 - 1/4Q^2$	$Q[1 + 1/(8Q^2)]$

amplitude is close to being Q times the static displacement for the same F_0, and it occurs at a frequency quite close to ω_0. At the frequency ω_0 itself the amplitude is precisely QA_0.

Figure 4–9 demonstrates how the sharpness of tuning of a resonant system varies with Q. The arrangement of an array of pendulums, as in Fig. 4–8(a), can be used to display the phenomenon. The Q can be increased, without changing ω_0, by making the bobs of the driven pendulums more massive. Figure 4–10 shows time-exposure photographs of the pendulums, first unloaded and then with two different degrees of loading. This clearly reveals the improvement in sharpness of tuning, even though the absolute amplitudes of oscillation in the three pictures are not strictly comparable. An instantaneous flash photograph is superimposed on each time-exposure photograph, displaying



(a) (b) (c)

Fig. 4–10 Time exposure photograph of Barton's pendulums (cf. Fig. 4–8) showing resonance properties. The pendulum bobs were light styrofoam spheres (from PSSC Electrostatics Kit). (a) Pendulum bobs unloaded and therefore heavily damped, showing little selective resonance. (b) Each pendulum bob lightly loaded (with one thumbtack) giving moderate damping and more selective resonance. (c) Each pendulum bob heavily loaded (one thumbtack + one small washer) giving small damping and fairly high Q. (Photos by Jon Rosenfeld, Education Research Center, M.I.T.) In each case an instantaneous flash photograph is superimposed in order to display the phase relationships among the driven pendulums.

the phase relationships among the driven pendulums for different Q, corresponding to Fig. 4–9(b).

TRANSIENT PHENOMENA

Our discussion so far has taken the steady state as being completely established, as if the driving force $F_0 \cos \omega t$ had been acting since far back in the past and all trace of any natural vibrations of the driven system had vanished. But of course in any real situation the driving force is first brought into action at some instant—which failing any reason to the contrary we might as well call $t = 0$—and it is only some time later that our steady-state conditions supervene. This transient stage may occupy a very long time indeed if the damping of the free vibrations is extremely small, and we shall even begin (again because of its mathematical simplicity) with the case in which the damping is effectively zero.

To make the problem quite explicit, let us suppose that we have a mass–spring system which, up to $t = 0$, is at rest. At $t = 0$ the driving force is turned on, and thereafter the motion is governed by Eq. (4–1), which we introduced at the beginning of this chapter:

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega t$$

or

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \tag{4–16}$$

Now we have already seen how this differential equation of the forced motion leads to the following equation for x:

$$x = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t \tag{4–17}$$

This equation, however, contains no adjustable constants of integration; the solution is completely specified by the values of m, ω_0, F_0, and ω. After our remarks in Chapter 3 about the need to introduce two constants of integration in solving a second-order differential equation, you may have wondered what became of them in this case. More specifically and, as it were, empirically, we can look at what Eq. (4–17) would give us for $t = 0$, the instant at which, according to our present assumptions, the driving force is first switched on. The result is impossible! If, for example, we suppose $\omega < \omega_0$, the displacement at $t = 0$ immediately assumes a positive value. But no system with nonzero inertia, acted on by a finite force, can be displaced through a nonzero distance in zero time. And if we suppose $\omega > \omega_0$, the result is a still greater absurdity—the mass would suddenly move to a *negative* displacement under the action of a *positive* force. Quite clearly Eq. (4–17) does not tell the whole story, and it is the transient that comes to the rescue.

Mathematically, the situation is this. Suppose that we have found a solution—call it x_1—to Eq. (4–16) so that

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 = \frac{F_0}{m} \cos \omega t$$

And now suppose that we have also found a solution—call it x_2—to the equation of free vibration, so that

$$\frac{d^2x_2}{dt^2} + \omega_0^2 x_2 = 0$$

Then by simple addition of these two equations we have

$$\frac{d^2(x_1 + x_2)}{dt^2} + \omega_0^2(x_1 + x_2) = \frac{F_0}{m}\cos\omega t$$

Thus the combination $x_1 + x_2$ is just as much a solution of the equation of forced motion as is x_1 alone. We have no mathematical reason to exclude the contribution from x_2; on the contrary, we are absolutely obliged to include it if we are to take care of the conditions existing at $t = 0$. We can say much the same thing, although less precisely, from a purely physical standpoint. The oscillations resulting from a brief impulse given to the system at $t = 0$ would certainly possess the natural frequency ω_0. It is only if a periodic force is applied over many cycles that the system learns, as it were, that it should oscillate with some different frequency ω. Thus one should expect that the motion, at least in its initial stages, contains contributions from both frequencies.

Turning now to the precise equations, the equation of the free vibration of frequency ω_0 *does* contain two adjustable constants—an amplitude and an initial phase. Let us call them B and β because we are using them to fit conditions at the beginning of the forced motion. Then, according to the ideas outlined above, we propose that the complete solution of the forced-motion equation is as follows:

$$x = B\cos(\omega_0 t + \beta) + C\cos\omega t \qquad (4\text{–}18)$$

where

$$C = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

We can now tailor Eq. (4–18) to fit the initial conditions (in this case) that $x = 0$ and $dx/dt = 0$ at $t = 0$. For the condition on x itself we have

$$0 = B\cos\beta + C$$

Also, differentiating Eq. (4–18), we have

$$\frac{dx}{dt} = -\omega_0 B\sin(\omega_0 t + \beta) - \omega C\sin\omega t$$

Hence, at $t = 0$, we have

$$0 = -\omega_0 B\sin\beta$$

The second condition requires that $\beta = 0$ or π. Taking the former (the final result is the same in either case) we get $B = -C$, so that Eq. (4–18) becomes

$$x = C(\cos \omega t - \cos \omega_0 t) \qquad (4\text{–}19)$$

which is a typical example of beats, as shown in Fig. 4–11(a). In the complete absence of damping these beats would continue indefinitely; no steady state corresponding to Eq. (4–17) alone would ever be reached. It is perhaps worth noting that the conditions just after $t = 0$ now make excellent sense. If ωt, $\omega_0 t \ll 1$, we can put

$$\cos \omega t \approx 1 - \frac{\omega^2 t^2}{2}$$

$$\cos \omega_0 t \approx 1 - \frac{\omega_0^2 t^2}{2}$$

Therefore,

$$x \approx \frac{F_0/m}{\omega_0^2 - \omega^2} \frac{(\omega_0^2 - \omega^2)t^2}{2} = \frac{1}{2} \frac{F_0}{m} t^2$$

Thus, precisely as we should expect, before the restoring forces have been called into play the mass starts out in the direction of the applied force with acceleration F_0/m.

You may wonder whether, granted that Eq. (4–18) can be justified as *a* solution of the forced-motion equation, it is therefore *the* solution. Here we shall merely assert that there is a uniqueness theorem for such differential equations, and if we have found any solution with the requisite number of adjustable constants, it is indeed the only solution of the problem.[1]

Turning now to the more realistic case in which damping is assumed to be present, we can without more ado postulate the following combination of free and steady-state motions:

$$x = Be^{-\gamma t/2} \cos(\omega_1 t + \beta) + A \cos(\omega t - \delta) \qquad (4\text{–}20)$$

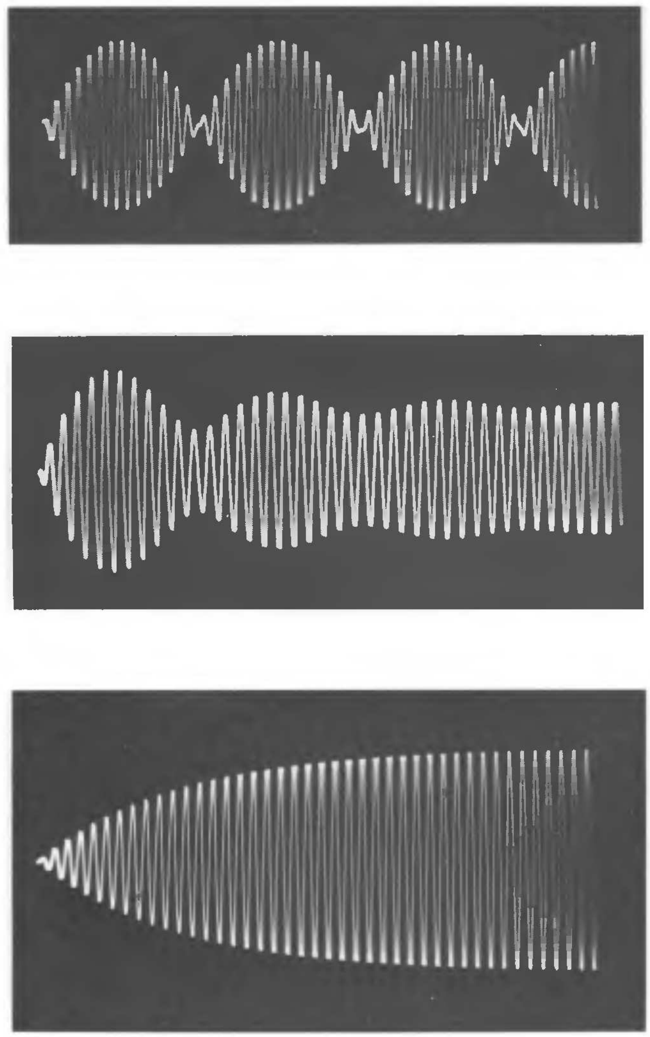
where

$$\omega_1 = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}$$

and A, δ are given by Eq. (4–11).

We shall not attempt here to delve into the purely mathematical details of fitting the values of B and β to the values of x and dx/dt at $t = 0$. It is just a more complicated version of what we did above for the undamped oscillator. In Fig. 4–11(b), however, we show the kind of motion that occurs—in general

[1]For a fuller discussion see, for example, W. T. Martin and E. Reissner, *Elementary Differential Equations*, Addison-Wesley, Reading, Mass., 2nd ed., 1961.



(a)
(b)
(c)

Fig. 4–11 (a) Response of an undamped harmonic oscillator to a periodic driving force, as described by Eq. (4–19). This beat pattern would continue indefinitely. (b) Transient behavior of a damped oscillator with a periodic driving force off resonance. (c) Transient behavior at exact resonance, showing smooth growth toward steady amplitude. (Photos by Jon Rosenfeld, Education Research Center, M.I.T.)

what looks like an attempt at beats, settling down to a motion of constant amplitude at the driving frequency ω. Figure 4–11(c) shows the much simpler transient effect that occurs when the damped oscillator is driven at its own natural frequency.

THE POWER ABSORBED BY A DRIVEN OSCILLATOR

It will often be a matter of importance and interest to know at what rate energy must be fed into a driven oscillator to maintain its oscillations at a fixed amplitude. As in any other dynamical situation, we can calculate the instantaneous power input, P, as the driving force times the velocity:

$$P = \frac{dW}{dt} = F\frac{dx}{dt} = Fv$$

Once again, let us consider first the undamped oscillator, for which (because there are no dissipative effects) the *mean* power input must come out to be zero. Taking the equations already developed, and assuming the steady-state solution, we have

$$F = F_0 \cos \omega t$$

$$x = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t = C \cos \omega t$$

Therefore,

$$v = -\omega C \sin \omega t$$

$$P = -\omega C F_0 \sin \omega t \cos \omega t$$

This power input, being proportional to $\sin 2\omega t$, is positive half the time and negative for the other half, averaging out to zero over any integral number of half-periods of oscillation. That is, energy is fed into the system during one quarter-cycle and is taken out again during the next quarter-cycle.

Coming now to the forced oscillator with damping, we have

$$x = A \cos(\omega t - \delta)$$

Therefore,

$$v = -\omega A \sin(\omega t - \delta)$$

We can write this as

$$v = -v_0 \sin(\omega t - \delta)$$

where v_0 is the maximum value of v for any given values of F_0 and ω. Taking the value of A from Eq. (4–14) we have

$$v_0(\omega) = \frac{F_0\omega_0/k}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}} \qquad (4\text{–}21)$$

The value of v_0 passes through a maximum at $\omega = \omega_0$, exactly, a phenomenon that we can call velocity resonance.

Now let us consider the work and the power needed to maintain the forced oscillations. We have

$$P = -F_0 v_0 \cos \omega t \sin(\omega t - \delta)$$
$$= -F_0 v_0 \cos \omega t(\sin \omega t \cos \delta - \cos \omega t \sin \delta)$$

i.e.,

$$P = -(F_0v_0 \cos\delta)\sin\omega t\cos\omega t + (F_0v_0\sin\delta)\cos^2\omega t \qquad (4\text{–}22)$$

If we average the power input over any integral number of cycles the first term in Eq. (4–22) gives zero. The average of $\cos^2 \omega t$, however, is $\frac{1}{2}$,[1] so that the average power input is given by

$$\overline{P} = \tfrac{1}{2}F_0v_0\sin\delta = \tfrac{1}{2}\omega AF_0\sin\delta$$

With the help of Eqs. (4–14) and (4–21) this becomes

$$\overline{P}(\omega) = \frac{F_0{}^2\omega_0}{2kQ}\,\frac{1}{\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}} \qquad (4\text{–}23)$$

We see that this power input, like the velocity, passes through a maximum at *precisely* $\omega = \omega_0$ for any Q. The maximum power is given by

$$P_m = \frac{F_0{}^2\omega_0 Q}{2k} = \frac{QF_0{}^2}{2m\omega_0} \qquad (4\text{–}24)$$

The dependence of $\overline{P}$ on ω for various Q is shown in Fig. 4–12(a). It may be noted that the power input drops off toward zero for very low and very high frequencies, and that except for low Q the curves are nearly symmetrical about the maximum. *It is convenient to define a width for these power resonance curves by taking the difference between those values of ω for which the power input is half of the maximum value.* This can be done in a particularly clear and useful way if (as in most cases of interest) Q is large. This means that the resonance is effectively contained within a narrow band of frequencies close to ω_0. It is then possible to write an approximate form of the equation for $\overline{P}(\omega)$, based on the following piece of algebra:

$$\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} = \frac{\omega_0{}^2 - \omega^2}{\omega\omega_0} = \frac{(\omega_0 + \omega)(\omega_0 - \omega)}{\omega\omega_0}$$

Hence, if $\omega \approx \omega_0$, we can put

$$\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \approx \frac{2\omega_0(\omega_0 - \omega)}{\omega_0{}^2} = \frac{2(\omega_0 - \omega)}{\omega_0}$$

Substituting this in the denominator of Eq. (4–23), we have

[1]Recall, for example, that $\cos^2\omega t = \frac{1}{2}(1 + \cos 2\omega t)$ and that $(\cos 2\omega t)_{av} = 0$ over a complete cycle.

$$\bar{P}(\omega) = \frac{F_0^2\omega_0}{2kQ}\,\frac{1}{\dfrac{4(\omega_0-\omega)^2}{\omega_0^2}+\dfrac{1}{Q^2}}$$

$$= \frac{F_0^2(\omega_0/Q)}{2(k/\omega_0^2)}\,\frac{1}{4(\omega_0-\omega)^2+(\omega_0/Q)^2}$$

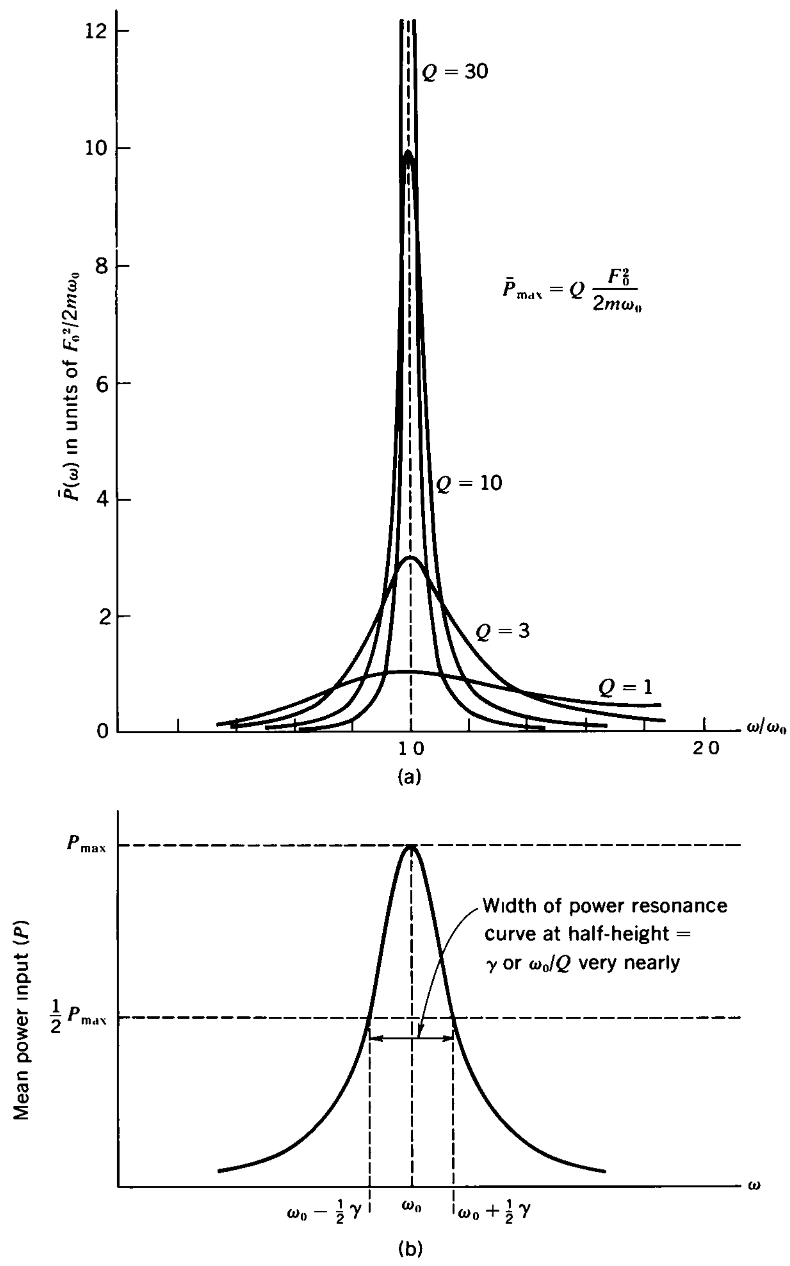


Fig. 4-12 (a) Mean power absorbed by a forced oscillator as a function of frequency for different values of Q. (b) Sharpness of resonance curve determined in terms of power curve.

Now we have met the quantity ω_0/Q before. It is the damping constant γ ($= b/m$) which characterizes the rate at which the energy of a damped oscillator was found to decay in the absence of a driving force:

$$E = E_0 e^{-(\omega_0/Q)t} = E_0 e^{-\gamma t} \tag{4–25}$$

[see Eq. (3–36)]. Thus the above equation for $\overline{P}$ can be written (remembering also that $k = m\omega_0{}^2$) in the following simplified form:

$$\text{(approximate) } \overline{P}(\omega) = \frac{\gamma F_0{}^2}{2m} \frac{1}{4(\omega_0 - \omega)^2 + \gamma^2} \tag{4–26}$$

The frequencies $\omega_0 \pm \Delta\omega$ at which $\overline{P}(\omega)$ falls to half of the maximum value $\overline{P}(\omega_0)$ are thus defined by

$$4(\Delta\omega)^2 = \gamma^2$$

i.e.,

$$2\,\Delta\omega \approx \frac{\omega_0}{Q} \tag{4–27}$$

Thus we find that the width of the resonance curve for the *driven* oscillator, as measured by the power input [Fig. 4–12(b)], is equal to the reciprocal of the time needed for the *free* oscillations to decay to $1/e$ of their initial energy. We can thus predict that if a system is observed to have a very narrow resonance response (as measured either by amplitude or by power absorption), then the decay of its free oscillations will be very slow. And conversely, of course, an observation of whether the free oscillations decay quickly or slowly will tell us whether the response of the driven oscillator is broad or narrow. What is our criterion of "slow" or "fast," "broad" or "narrow"? Equations (4–26) and (4–27) tell us the answer. We can say that the resonance is narrow if the width is only a small fraction of the resonant frequency, i.e., if

$$\frac{2\,\Delta\omega}{\omega_0} \ll 1 \tag{4–28a}$$

and we can say that the decay of free oscillations is slow if the oscillator loses only a small fraction of its energy in one period of oscillation. Now from Eq. (4–25) we have

$$\frac{\Delta E}{E} \approx -\gamma\,\Delta t$$

If for Δt we put the time $2\pi/\omega_0$, which is approximately equal to the period of the free damped oscillation [Eq. (3–40)], we have

$$\frac{\Delta E}{E} \approx -\frac{2\pi\gamma}{\omega_0}$$

Thus a slow decay means

$$\frac{2\pi\gamma}{\omega_0} \ll 1 \tag{4–28b}$$

Since $\gamma = 2\,\Delta\omega = \omega_0/Q$, the conditions described by Eqs. (4–28a) and (4–28b) can both be expressed by saying that the dimensionless quantity Q must be large.

This relation between the resonance width of forced oscillation and the decrement of free oscillations is characteristic of a wide variety of oscillatory physical systems, not only the mechanical oscillator which we are here using as an example. In fact, whenever such a physical system, in free oscillation, shows an exponential loss of energy with time, it also displays a driven response having resonance characteristics.

EXAMPLES OF RESONANCE

In the course of our discussions we have made passing references to the fact that many systems which, on the face of it, have very little in common with a mass on a spring, nevertheless exhibit a similar resonance behavior. In concentrating on the behavior of a simple mechanical system, however, our analysis became very detailed and specific. Now we shall broaden our view again, and say something about resonance in quite different systems.

If we are to extend our ideas in this way, we need to be able to say in rather general terms what we mean by resonance, and we can begin by asking ourselves: What is the real essence of the behavior of the mass and spring system? And putting aside the mathematics we can say this: The system is acted on by an external agency, one parameter of which (the frequency) is varied. The response of the system, as measured by its amplitude and phase, or by the power absorbed, undergoes rapid changes as the frequency passes through a certain value. The form of the response is described by two quantities—a frequency ω_0 and a width γ $(= \omega_0/Q)$—which characterize the distinctive properties of the driven system. Resonance is the phenomenon of driving the system under such conditions that the interaction between the driving agency and the system is maximized. Whatever the particular criteria applied, one can say that the interaction has its maximum at or near ω_0, and that its most marked changes occur over a range of about $\pm\gamma$ with respect to the maximum.

When we carry over these ideas to the resonance behavior of other physical systems, we shall find that the quantities that characterize a resonance are not always frequency, absorbed power, and amplitude. This will appear in some of the examples that we shall now discuss.

ELECTRICAL RESONANCE

One of the most familiar and important resonant systems is the electrical system made up of a capacitor and a coil, as shown in Fig. 4–13. The analysis of such a system has a remarkable similarity to the mechanical systems with which we have been concerned so far. Let us consider first the free oscillations, ignoring for the moment any dissipative process associated with the electrical resistance. To begin with, we shall briefly describe the essential electrical behavior of the individual components.

The capacitor is a device for storing electric charge and the associated electrostatic potential energy. Its capacitance C is defined as the measure of the charge q applied to the capacitor plates divided by the measure of the voltage difference that this charge produces:

$$C = \frac{q}{V_C}$$

Therefore,

$$V_C = \frac{q}{C}$$

The action of the coil requires a somewhat more detailed description. Under D-C conditions the coil offers no opposition to the flow of current, but if the current is changing with time it is found that the coil (which we shall henceforth call an inductor) acts to oppose that change (Lenz's law). Under these circumstances

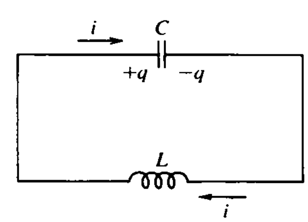


Fig. 4–13 Capacitor and inductor in series: the basic electrical resonance system.

there is a voltage difference V_L between the ends of the inductor, and this voltage is proportional to the rate of change of the current i. The inductance L is defined by the relation

$$V_L = L\frac{di}{dt}$$

This equation says that a voltage V_L must be applied between the ends of the inductor in order to make the current change at the rate di/dt.

In a circuit made up of just these two components, the sum of V_C and V_L must be zero, because an imaginary journey through the capacitor and then through the inductor brings us back to the same point on the circuit. Thus we have

$$\frac{q}{C} + L\frac{di}{dt} = 0 \qquad (4\text{–}29)$$

Now there is an intimate connection between q and i, because the current in the circuit is just the rate of flow of charge past any point. A current i flowing for a time dt in the wire connected to a capacitor plate will increase the charge on that plate by the amount $dq = i\,dt$, so we have

$$i = \frac{dq}{dt}$$

$$\frac{di}{dt} = \frac{d^2q}{dt^2}$$

Hence Eq. (4–29) can be written

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \qquad (4\text{–}30)$$

But this is precisely like the basic differential equation of SHM for a mass–spring system, with q playing the role of x, L appearing in the place of m, and $1/C$ replacing the spring constant k. We can confidently assume the existence of free electrical oscillations such that

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Now let us consider the effect of introducing a resistor, of resistance R, as in Fig. 4–14(a). At current i it is necessary to have a voltage V_R ($= iR$) applied between the ends of the resistor. Thus the statement of zero net voltage drop in one complete tour of the circuit is as follows:

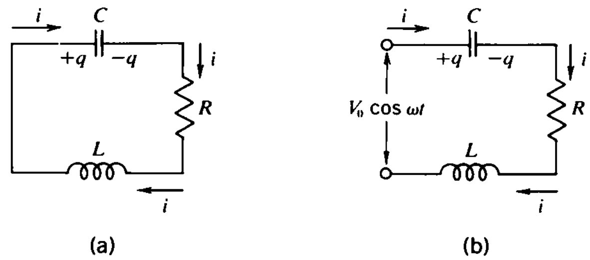


Fig. 4–14 (a) Capacitor, inductor, and resistor in series. (b) Capacitor, inductor, and resistor in series driven by a sinusoidal voltage.

$$\frac{q}{C} + iR + L\frac{di}{dt} = 0$$

i.e.,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

or

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0 \tag{4–31}$$

In this equation, R/L plays exactly the role of the damping constant γ, and in such a circuit the charge on the capacitor plates (and the voltage V_C) will undergo exponentially damped harmonic oscillations.

Finally, if the circuit is driven by an alternating applied voltage, we have a typical forced-oscillator equation:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{V_0}{L}\cos\omega t \tag{4–32}$$

Compare:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t \tag{4–33}$$

The connection between Eqs. (4–32) and (4–33) becomes even closer if one considers the energy of the system. Just as $F\,dx$ is the amount of work done by the driving force F in a displacement dx, so $V\,dq$ is the amount of work done by the driving voltage V when an amount of charge dq passes through the circuit. One can regard the oscillation as involving the periodic transfer of energy between the capacitor and the inductor, with a continual dissipation of energy in the resistor. Comparison of the mechanical and electrical equations suggests the classification of analogous quantities, as shown in Table 4–2.

We have discussed this phenomenon of electrical resonance

TABLE 4-2: MECHANICAL AND ELECTRICAL RESONANCE PARAMETERS

Mechanical system	*Electrical system*
Displacement x	Charge q
Driving force F	Driving voltage V
Mass m	Inductance L
Viscous force constant b	Resistance R
Spring constant k	Reciprocal capacitance $1/C$
Resonant frequency $\sqrt{k/m}$	Resonant frequency $1/\sqrt{LC}$
Resonance width $\gamma = b/m$	Resonance width $\gamma = R/L$
Potential energy $\frac{1}{2}kx^2$	Energy of static charge $\frac{1}{2}q^2/C$
Kinetic energy $\frac{1}{2}m(dx/dt)^2 = \frac{1}{2}mv^2$	Electromagnetic energy of moving charge $\frac{1}{2}L(dq/dt)^2 = \frac{1}{2}Li^2$
Power absorbed at resonance $F_0{}^2/2b$	Power absorbed at resonance $V_0{}^2/2R$

at some length because of its extremely close likeness to mechanical resonance. Our other examples, although of great physical importance, do not fall so completely into this pattern, and we shall dispose of them more briefly.

OPTICAL RESONANCE

We have a great wealth of evidence that atoms behave like sharply tuned oscillators in the processes of emitting and absorbing light. Whenever the emission of light occurs under such conditions that the radiating atoms are effectively isolated from each other, as in a gas at low pressure, the spectrum consists of discrete, very narrow lines; i.e., the radiated energy is concentrated at particular wavelengths. An incandescent solid—e.g., the filament of a light bulb—emits a continuous spectrum, but the situation here is quite different, because each atom in a solid is strongly linked to its neighbors, causing a drastic change in the dynamical state of the electrons chiefly responsible for visible or near-visible radiation.

We have just spoken of atoms as oscillators that emit their characteristic frequencies. But how does this fit in with the photon description of radiation, and with the picture of the radiative process as one in which the atom undergoes a quantum jump? The answer is by no means obvious. Before the advent of quantum theory, one could visualize an electron describing a circular orbit within an atom, and emitting light of a frequency equal to its own orbital frequency. But now we can only say that the frequency of the light is defined (through $E = h\nu$) by the energy difference between two states of the atom; we can no longer identify that frequency with a vibration of the atom itself. Nevertheless the concept of the atom as an oscillator does in some respects survive. If the emitted light is analyzed with an interferometer, it is found to consist of wave trains of finite length. The length of the wave trains, divided by c, defines a time τ which corresponds to the mean life of the radiating atoms in their excited state, and the surplus energy of a collection of excited atoms decays exponentially as $e^{-t/\tau}$ ($= e^{-\gamma t}$) as the energy is radiated away. Neither the photon picture nor the wave picture alone tells us the whole story, but the model of the atom as a damped oscillator provides an acceptable description of some important aspects of the radiative process.

As we have seen, the concomitant of a natural frequency of free oscillation is a resonance absorption at about that same frequency. In the case of visible light the frequencies are too high ($\approx 10^{15}$ Hz) to be measurable, but we are able to describe both emission and absorption in terms of characteristic wavelengths. Probably the most famous example of resonance absorption for light is provided by the Fraunhofer lines. These are the dark lines that are observed in a spectrum analysis of the sun; they are named after Joseph von Fraunhofer, who in a careful study mapped 576 of them in 1814. Figure 4–15(a) shows a portion of

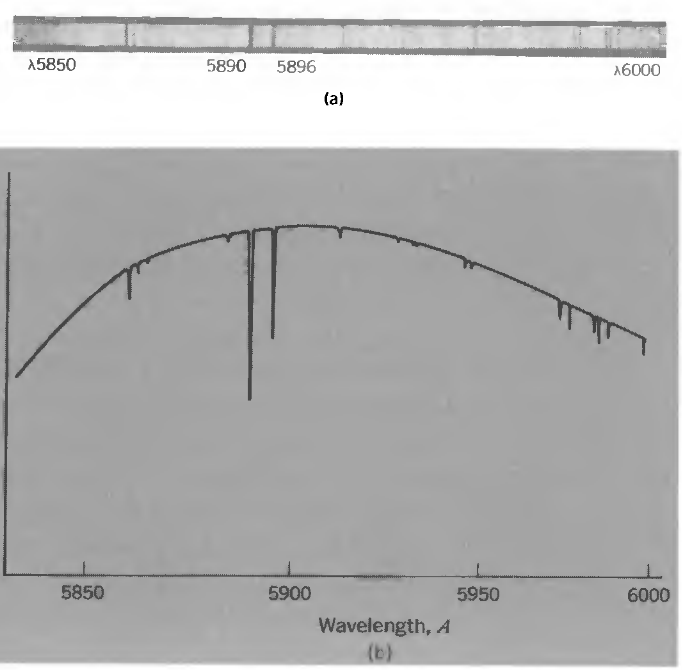


Fig. 4–15 (a) Portion of the solar spectrum, showing the famous sodium D lines at 5890 and 5896 Å. (From F. A. Jenkins and H. E. White, Fundamentals of Optics, *McGraw-Hill, New York, 1957.) (b) A qualitative representation of the intensity of the solar spectrum as a function of wavelength, over the range shown in (a).*

the solar spectrum; the prominent Fraunhofer lines at 5890 and 5896 Å are due to sodium. Figure 4–15(b) shows qualitatively what a plot of intensity versus wavelength looks like; the intensity dips sharply at the wavelength of the Fraunhofer lines, but is not zero. (It was not Fraunhofer who first observed the absorption lines,[1] but it was he who first recognized that some of them coincided in wavelength with bright emission lines produced by laboratory sources. It remained, however, for Kirchhoff and Bunsen in 1861 to make a detailed comparison of the solar spectrum with the arc and spark spectra of pure elements.)

One can be sure that the Fraunhofer lines are the result of resonance absorption processes. The picture is that the continuous radiation from hot and relatively dense matter near the sun's surface is selectively filtered, as it passes outward, by atoms in the more tenuous vapors of the solar atmosphere. It would be satisfying if one could trace out the detailed shape of an optical absorption line and relate its width to the characteristic time ($= 1/\gamma$) for the decay of the spontaneous emission. This, however, is extremely hard to do. The chief enemy is the Doppler effect. Both direct and indirect evidence show that a typical lifetime for an excited atom emitting visible light is about 10^{-8} sec, so that γ is about $10^8\ \text{sec}^{-1}$. The angular frequency of the emitted light, as defined by $2\pi c/\lambda$, is about $4 \times 10^{15}\ \text{sec}^{-1}$. Thus we can calculate a line width $\delta\lambda$ as follows:

$$\frac{\delta\lambda}{\lambda} \approx \frac{\delta\omega}{\omega_0} = \frac{\gamma}{\omega_0} \approx \frac{10^8}{4 \times 10^{15}} \approx 2 \times 10^{-8}$$

(Hence $\delta\lambda \approx 10^{-4}$Å for $\lambda \approx 5000$ Å.) But, unless special precautions are taken, the emitting atoms have random thermal motions of several hundred meters per second, and we can estimate a Doppler broadening of the spectral lines:

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \approx 10^{-6}$$

The Doppler effect is thus about 100 times greater than any effect due to the true lifetime of the radiating atom. Interatomic collisions also disturb the situation, so that the resonance shapes of spectral lines are more a matter of inference than of direct spectroscopic observation.

[1]They were first noted by W. H. Wollaston in 1802. By 1895 a classic study by the American physicist H. A. Rowland had resulted in the mapping of 1100 of them. Today about 26,000 lines have been catalogued between 3000 and 13,000 Å.

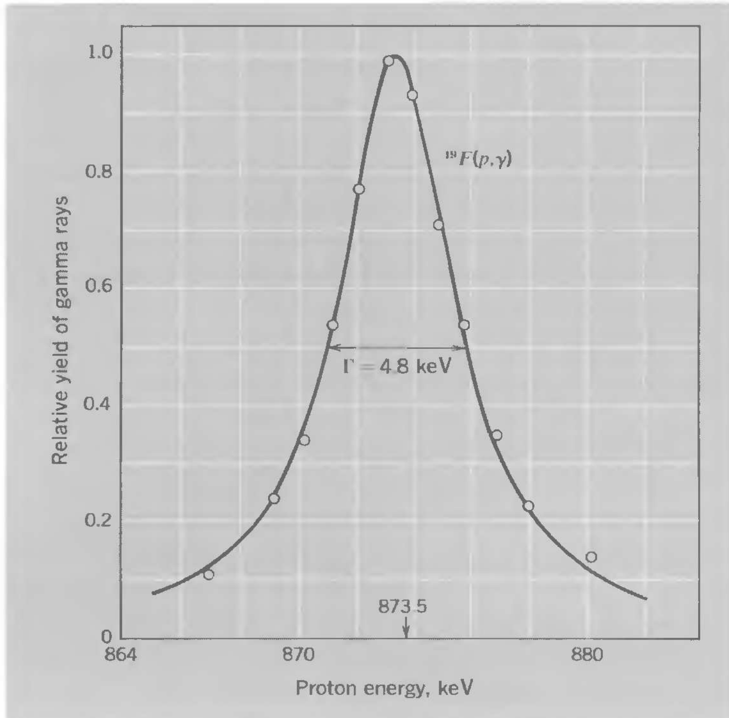


Fig. 4-16 Yield of gamma rays as a function of the energy of bombarding protons in the reaction $p + {}^{19}F \rightarrow {}^{20}Ne + \gamma$. [*From data of R. G. Herb, S. C. Snowden, and O. Sala,* Phys. Rev., *75, 246 (1949).*]

NUCLEAR RESONANCE

The literature of nuclear physics contains innumerable examples of nuclear resonances; Fig. 4-16 shows one of them. This process of nuclear resonance differs in several ways from anything we have discussed so far. The subject of Fig. 4-16 is a nuclear reaction; the graph shows the relative yield of gamma rays as a target of fluorine is bombarded with protons of different energies around 875 keV. But what is the resonant system? It is not the bombarded fluorine but the compound nucleus—^{20}Ne in an excited state, denoted ^{20}Ne*, formed when a fluorine nucleus captures a proton. This compound nucleus is unstable, and one of its decay modes is by emission of gamma rays. The complete process can be written as follows:

$$^{1}_{1}H + {}^{19}_{9}F \rightarrow {}^{20}_{10}Ne^* \rightarrow {}^{20}_{10}Ne + \gamma$$

(The subscript shows the number of protons in a nucleus, and the superscript the total of protons plus neutrons.)

The controllable parameter—the independent variable of the interaction—is not a frequency but the energy of the bombarding proton. This defines a basic property of the resonance: the total energy of the $^{20}Ne^*$ in its rest frame. The response of the system is measured, not in terms of amplitude or absorbed power, but in terms of the probability that an incident proton will cause a gamma ray to be produced. This probability can be described in terms of the effective target area (or cross section, σ) that each fluorine nucleus presents to the incident proton beam. Finally, the detailed shape of the resonance curve is very similar in analytic form to the approximate form (for high Q) of the absorbed power curve of a mechanical oscillator [Eq. (4–26) and Fig. 4–12]. A nuclear resonance such as the one of Fig. 4–16 can be well described by the equation

$$\sigma(E) = \frac{\sigma(E_0)}{\dfrac{4(E_0 - E)^2}{\Gamma^2} + 1} \tag{4–34}$$

The energy E_0 then corresponds to the peak of the resonance curve, and the total width of the curve at half-height is given by Γ. Defined in this way, the energy width Γ is strictly analogous to the frequency width γ of a mechanical or electrical resonance. The full curve in Fig. 4–16 is drawn according to Eq. (4–34) with appropriate values of E_0 and Γ, and it can be seen that the fit to the data is excellent.

NUCLEAR MAGNETIC RESONANCE

As a last example of resonance in other fields of physics, we shall mention the resonant process by which atomic nuclei, behaving as tiny magnets, can be flipped over in a magnetic field. It depends upon a quantum phenomenon: that atomic magnets are limited to having only a few discrete possible orientations with respect to a magnetic field in a given direction. A proton, to take a specific example, has only two possible orientations, one corresponding roughly to the north-seeking orientation of an ordinary compass needle, and the other corresponding to the reverse of this. There is a well-defined energy difference between these orientations, corresponding to the work done against the magnetic forces in turning the nuclear magnet from one position to the other. This energy difference is directly proportional to the strength of the magnetic field in which the proton finds itself. If photons of just the right energy come along, they can cause the protons to switch from one orientation to the other. This can be brought about by injecting electromagnetic radiation of just the right frequency; for protons in a field of about 5000 G the resonance frequency is about 21 MHz. If all the protons in about 1 cm^3 of water are flipped in this way, they can be made to produce (through electromagnetic induction) a readily detectable voltage in a pickup coil. If the magnetic field were held constant, one would see this signal as a resonant function of the frequency of the injected radiation. It is much more convenient, however, to use a constant, sharply defined radiofrequency and vary the strength of the applied magnetic field B. The magnitude of the nuclear magnetic resonance signal can then be expressed as a resonant function of the field strength:

$$V(B) = \frac{V_0}{\dfrac{4(B_0 - B)^2}{(\Delta B)^2} + 1} \tag{4-35}$$

where B_0 is the field strength at exact resonance and ΔB is the width of the resonance at half-height.

For their quite independent research on this phenomenon,

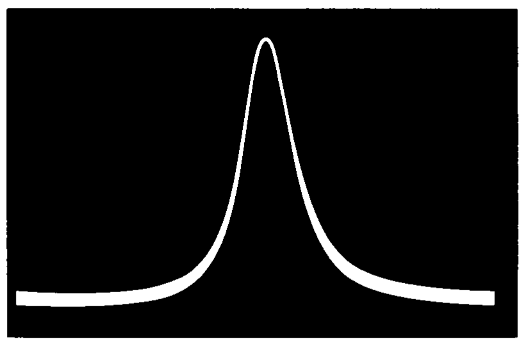


Fig. 4–17 Magnetic resonance line of protons in water containing $MnSO_4$ as a paramagnetic catalyst and obtained from that component of the nuclear induction signal which corresponds to absorption. The photograph is of the trace on a cathode-ray oscillograph with the vertical deflection *arising from the rectified and amplified signal and the* horizontal deflection *corresponding to different values of the constant field. From* Nobel Lectures: Physics (1942–1962), *Elsevier, Amsterdam, 1964.*

F. Bloch and E. M. Purcell shared the Nobel Prize in physics in 1952. Figure 4–17 comes from the Nobel lecture that Bloch gave at that time.

ANHARMONIC OSCILLATORS

So far this chapter reads altogether too much like a success story. Everything works. We write down a differential equation and obtain in every case an analytic solution that fits it exactly. We point to actual physical systems that apparently conform perfectly to our very simple mathematical model. Is nature really so accommodating? The answer is that in certain cases—numerous and varied enough to be of great physical importance—a system can indeed be represented, with impressive accuracy, as a damped oscillator with a restoring force proportional to the displacement and a resistive force proportional to the velocity. But this is an astonishing stroke of luck, and we have in fact been treading a very narrow path. To appreciate just how special and favorable are the situations that we have discussed, we shall glance briefly at the effect of modifying the equations of motion.

Our original equation for the free oscillation of a mass on a spring without damping was the following:

$$F = m\frac{d^2x}{dt^2} = -kx$$

This holds if the spring obeys a linear relation (Hooke's law) for any amount of extension or compression. But no real spring behaves quite like this. With many springs it takes a slightly different size of force to produce a given extension than to produce an equal compression. The simplest asymmetry of this kind is represented by a term in F proportional to x^2. Or it may be that the spring is symmetrical with respect to positive and negative displacements, but that there is not strict proportionality of F to x. The simplest symmetrical effect of this kind is described by a term in F proportional to x^3. The equations of motion for these cases can be written as follows:

$$\text{Nonlinear, asymmetric: } m\frac{d^2x}{dt^2} + kx + \alpha x^2 = 0 \qquad (4\text{–}36\text{a})$$

$$\text{Nonlinear, symmetric: } m\frac{d^2x}{dt^2} + kx + \beta x^3 = 0 \qquad (4\text{–}36\text{b})$$

If we try a solution of the form $x = A\cos\omega_0 t$ in either of the above equations we find at once that it does not work; the motion is no longer describable as a harmonic vibration at some unique frequency ω_0. We have instead what is called an anharmonic oscillator. The motion is still periodic, in that (assuming no damping) a given state of the motion recurs at equal intervals $T = 2\pi/\omega_0$, but instead of having $x = A\cos\omega_0 t$ we find that an infinite set of harmonics of ω_0 is now needed to describe the motion; i.e., we must put

$$x = \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \delta_n)$$

in order to have a form of x that will satisfy the differential equations.

In similar fashion, a resistive force varying as v^2 or v^3, instead of v, makes impossible a clean, simple analytic description of the motion of a damped oscillator.

What happens if an oscillator with nonlinear terms (in restoring force, damping force, or both) is subjected to a sinusoidal driving force? We shall not try to spell out the answer but leave it as a challenge for your spare moments. Take, for example, an oscillator whose free oscillations are described by Eq. (4–36a) with a pure viscous force ($\sim dx/dt$) added, and assume a driving force $F = F_0 \cos \omega t$. Assume $\alpha x^2 \ll kx$, put $k/m = \omega_0{}^2$, and see if you can determine the frequency or frequencies ω for which the system exhibits resonance behavior. After investigating this problem you will realize that the simple harmonic oscillator is well named, and you will appreciate why a physicist will use it as a model of a vibratory system if it can possibly be justified.

PROBLEMS

4–1 Construct a table, covering as wide a range as possible, of resonant systems occurring in nature. Indicate the order of magnitude of (a) the physical size of each system, and (b) its resonant frequency.

4–2 Consider how to solve the steady-state motion of a forced oscillator if the driving force is of the form $F = F_0 \sin \omega t$ instead of $F_0 \cos \omega t$.

4–3 An object of mass 0.2 kg is hung from a spring whose spring constant is 80 N/m. The body is subject to a resistive force given by $-bv$, where v is its velocity (m/sec) and b = 4 N-m^{-1} sec.

(a) Set up the differential equation of motion for free oscillations of the system, and find the period of such oscillations.

(b) The object is subjected to a sinusoidal driving force given by $F(t) = F_0 \sin \omega t$, where F_0 = 2 N and ω = 30 sec^{-1}. In the steady state, what is the amplitude of the forced oscillation?

4–4 A block of mass m is connected to a spring, the other end of which is fixed. There is also a viscous damping mechanism. The following observations have been made on this system:

(1) If the block is pushed horizontally with a force equal to mg, the static compression of the spring is equal to h.

(2) The viscous resistive force is equal to mg if the block moves with a certain known speed u.

(a) For this complete system (including both spring and damper) write the differential equation governing horizontal oscillations of the mass in terms of m, g, h, and u.

Answer the following for the case that $u = 3\sqrt{gh}$:

(b) What is the angular frequency of the damped oscillations?

(c) After what time, expressed as a multiple of $\sqrt{h/g}$, is the *energy* down by a factor $1/e$?

(d) What is the Q of this oscillator?

(e) This oscillator, initially in its rest position, is suddenly set into motion at $t = 0$ by a bullet of negligible mass but nonnegligible momentum traveling in the positive x direction. Find the value of the phase angle δ in the equation $x = Ae^{-\gamma t/2}\cos(\omega t - \delta)$ that describes the subsequent motion, and sketch x versus t for the first few cycles.

(f) If the oscillator is driven with a force $mg \cos \omega t$, where $\omega = \sqrt{2g/h}$, what is the amplitude of the steady-state response?

4-5 A simple pendulum has a length (l) of 1 m. In free vibration the amplitude of its swings falls off by a factor e in 50 swings. The pendulum is set into forced vibration by moving its point of suspension horizontally in SHM with an amplitude of 1 mm.

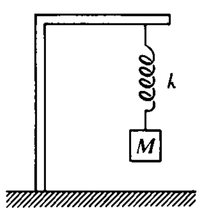
(a) Show that if the horizontal displacement of the pendulum bob is x, and the horizontal displacement of the support is ξ, the equation of motion of the bob for small oscillations is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \frac{g}{l}x = \frac{g}{l}\xi$$

Solve this equation for steady-state motion, if $\xi = \xi_0 \cos \omega t$. (Put $\omega_0^2 = g/l$.)

(b) At exact resonance, what is the amplitude of the motion of the pendulum bob? (First, use the given information to find Q.)

(c) At what angular frequencies is the amplitude half of its resonant value?



4-6 Imagine a simple seismograph consisting of a mass M hung from a spring on a rigid framework attached to the earth, as shown. The spring force and the damping force depend on the displacement and velocity relative to the earth's surface, but the dynamically significant acceleration is the acceleration of M relative to the fixed stars.

(a) Using y to denote the displacement of M relative to the earth and η to denote the displacement of the earth's surface itself, show that the equation of motion is

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2}$$

(b) Solve for y (steady-state vibration) if $\eta = C \cos \omega t$.

(c) Sketch a graph of the amplitude A of the displacement y as a function of ω (supposing C the same for all ω).

(d) A typical long-period seismometer has a period of about 30 sec and a Q of about 2. As the result of a violent earthquake the earth's surface may oscillate with a period of about 20 min and with an amplitude such that the maximum acceleration is about 10^{-9} m/sec^2. How small a value of A must be observable if this is to be detected?

4-7 Consider a system with a damping force undergoing forced oscillations at an angular frequency ω.

(a) What is the instantaneous kinetic energy of the system?

(b) What is the instantaneous potential energy of the system?

(c) What is the ratio of the average kinetic energy to the average potential energy? Express the answer in terms of the ratio ω/ω_0.

(d) For what value(s) of ω are the average kinetic energy and the average potential energy equal? What is the total energy of the system under these conditions?

(e) How does the total energy of the system vary with time for an arbitrary value of ω? For what value(s) of ω is the total energy constant in time?

4–8 A mass m is subject to a resistive force $-bv$ but *no* springlike restoring force.

(a) Show that its displacement as a function of time is of the form

$$x = C - \frac{v_0}{\gamma} e^{-\gamma t}$$

where $\gamma = b/m$.

(b) At $t = 0$ the mass is at rest at $x = 0$. At this instant a driving force $F = F_0 \cos \omega t$ is switched on. Find the values of A and δ in the steady-state solution $x = A \cos(\omega t - \delta)$.

(c) Write down the general solution [the sum of parts (a) and (b)] and find the values of C and v_0 from the conditions that $x = 0$ and $dx/dt = 0$ at $t = 0$. Sketch x as a function of t.

4–9 (a) A forced damped oscillator of mass m has a displacement varying with time given by $x = A \sin \omega t$. The resistive force is $-bv$. From this information calculate how much work is done against the resistive force during one cycle of oscillation.

(b) For a driving frequency ω *less* than the natural frequency ω_0, sketch graphs of potential energy, kinetic energy, and total energy for the oscillator over one complete cycle. Be sure to label important turning points and intersections with their values of energy and time.

4–10 The power input to maintain forced vibrations can be calculated by recognizing that this power is the mean rate of doing work against the resistive force $-bv$.

(a) Satisfy yourself that the instantaneous rate of doing work against this force is equal to bv^2.

(b) Using $x = A \cos(\omega t - \delta)$, show that the mean rate of doing work is $b\omega^2 A^2/2$.

(c) Substitute the value of A at any arbitrary frequency and hence obtain the expression for $\overline{P}$ as given in Eq. (4–23).

4–11 Consider a damped oscillator with $m = 0.2$ kg, $b = 4$ N-m^{-1} sec and $k = 80$ N/m. Suppose that this oscillator is driven by a force $F = F_0 \cos \omega t$, where $F_0 = 2$ N and $\omega = 30$ sec^{-1}.

(a) What are the values A and δ of the steady-state response described by $x = A \cos(\omega t - \delta)$?

(b) How much energy is dissipated against the resistive force in one cycle?

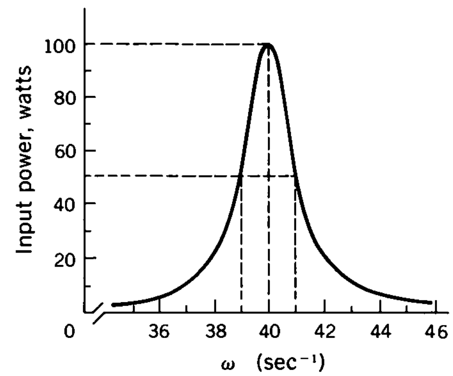
(c) What is the mean power input?

4–12 An object of mass 2 kg hangs from a spring of negligible mass. The spring is extended by 2.5 cm when the object is attached. The top end of the spring is oscillated up and down in SHM with an amplitude of 1 mm. The Q of the system is 15.

(a) What is ω_0 for this system?

(b) What is the amplitude of forced oscillation at $\omega = \omega_0$?

(c) What is the mean power input to maintain the forced oscillation at a frequency 2% greater than ω_0? [Use of the approximate formula, Eq. (4–26), is justified.]

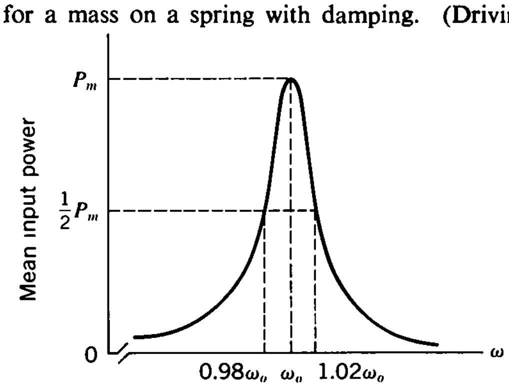


4–13 The graph shows the power resonance curve of a certain mechanical system when driven by a force $F_0 \sin \omega t$, where F_0 = constant and ω is variable.

(a) Find the numerical values of ω_0 and Q for this system.

(b) The driving force is turned off. After how many cycles of free oscillation is the energy of the system down to $1/e^5$ of its initial value? (e = 2.718.) (To a good approximation, the period of free oscillation can be set equal to $2\pi/\omega_0$.)

4–14 The figure shows the mean power input $\overline{P}$ as a function of driving frequency for a mass on a spring with damping. (Driving force =



$F_0 \sin \omega t$, where F_0 is held constant and ω is varied.) The Q is high enough so that the mean power input, which is maximum at ω_0, falls to half-maximum at the frequencies $0.98\omega_0$ and $1.02\omega_0$.

(a) What is the numerical value of Q?

(b) If the driving force is removed, the energy decreases according to the equation

$$E = E_0 e^{-\gamma t}$$

What is the value of γ?

(c) If the driving force is removed, what fraction of the energy is lost per cycle?

A new system is made in which the spring constant is doubled, but the mass and viscous medium are unchanged, and the same driving force $F_0 \sin \omega t$ is applied. In terms of the corresponding quantities for the original system, find the values of the following:

(d) The new resonant frequency ω_0'.

(e) The new quality factor Q'.

(f) The maximum mean power input $\overline{P}_m'$.

(g) The total energy of the system at resonance, E_0'.

4–15 The free oscillations of a mechanical system are observed to have a certain angular frequency ω_1. The same system, when driven by a force $F_0 \cos \omega t$ (where F_0 = const. and ω is variable), has a power resonance curve whose angular frequency width, at half-maximum power, is $\omega_1/5$.

(a) At what angular frequency does the maximum power input occur?

(b) What is the Q of the system?

(c) The system consists of a mass m on a spring of spring constant k. In terms of m and k, what is the value of the constant b in the resistive term $-bv$?

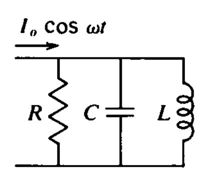
(d) Sketch the amplitude response curve, marking a few characteristic points on the curve.

4–16 For the electrical system in the figure, find

(a) The resonant frequency, ω_0.

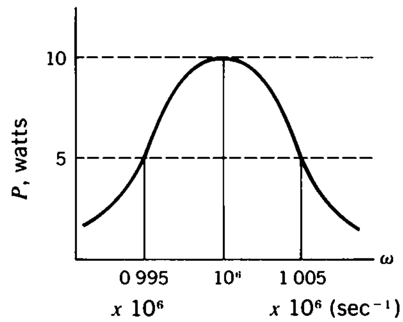
(b) The resonance width, γ.

(c) The power absorbed at resonance.



4–17 The graph shows the mean power absorbed by an oscillator when driven by a force of constant magnitude but variable angular frequency ω.

(a) At exact resonance, how much work per cycle is being done against the resistive force? (Period = $2\pi/\omega$.)



(b) At exact resonance, what is the *total* mechanical energy E_0 of the oscillator?

(c) If the driving force is turned off, how many seconds does it take before the energy decreases to a value $E = E_0e^{-1}$?

The question of the vibration of connected particles is a peculiarly interesting and important problem . . . it is going to have many applications.

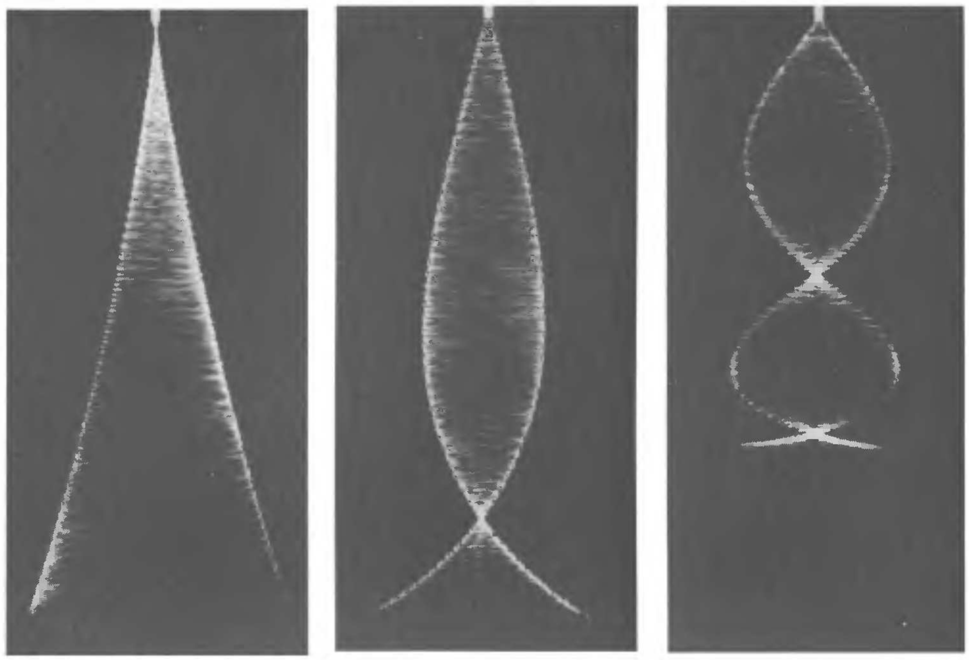
LORD KELVIN, *Baltimore Lectures* (1884)

5

Coupled oscillators and normal modes[1]

THROUGHOUT THE PRECEDING TWO CHAPTERS we have confined our analysis to systems having only one type of free vibration, and characterized by a single natural frequency. A real physical system, however, is usually capable of vibrating in many different ways, and may resonate to many different frequencies—like a sort of grand piano. We speak of these various characteristic vibrations as *modes*, or, for reasons that will emerge later, as *normal modes* of the system. A simple example is a flexible chain suspended from one end. It is found that there is a whole succession of frequencies at which every point on the chain vibrates in SHM at the same frequency, so that the shape of the chain remains constant in the sense that the displacements of the various parts always preserve fixed ratios. The first three modes (in ascending order of frequency) for such a chain are shown in Fig. 5–1. This is in effect only a one-dimensional object, and the variety of natural modes of oscillation for two- and three-dimensional objects is still greater.

[1]This whole chapter may be bypassed if it is preferred to proceed directly to the discussion of vibrations and waves in effectively continuous media. On the other hand, an acquaintance with the contents of the present chapter, even in rather general terms, may help in appreciating the sequel, for the many-particle system does provide the natural link between the single oscillator and the continuum. And it is not as mathematically formidable as it may appear at first sight.



(a) (b) (c)

Fig. 5–1 First three normal modes of vertical chain with upper end fixed. (The tension is provided at each point by the weight of the chain below that point and so increases linearly with distance from the bottom.)

How do we go about the job of accounting for these numerous modes and calculating their frequencies? The clue to this question lies in the fact that an extended object can be regarded as a large number of simple oscillators coupled together. A solid body, for example, is composed of many atoms or molecules. Every atom may behave as an oscillator, vibrating about an equilibrium position. But the motion of each atom affects its neighbors so that, in effect, all the atoms of the solid are coupled together. The question then becomes: How does the coupling affect the behavior of the individual oscillators?

We shall begin by discussing in some detail the properties of a system of just two coupled oscillators. The change from one oscillator to two may seem rather trivial, but this new system has some novel and surprising features. Moreover, in analyzing its behavior we shall develop essentially all the theoretical tools we need to handle the problem of an arbitrarily large number of coupled oscillators—which will be our ultimate concern. And this means that, from quite simple beginnings, we can end up with a significant insight into the dynamical properties of something as complicated as a crystal lattice. That is no small achievement, and it is worth the little extra amount of mathematical effort that our discussion will entail.

TWO COUPLED PENDULUMS

Let us begin with a very simple example. Take two identical pendulums, A and B, and connect them with a spring whose relaxed length is exactly equal to the distance between the pendulum bobs, as shown in Fig. 5–2. Draw pendulum A aside while holding B fixed and then release both of them. What happens?

Pendulum A swings from side to side, but its amplitude of oscillation continuously decreases. Pendulum B, initially undisplaced, gradually begins to oscillate and its amplitude continuously increases. Soon, A and B have equal amplitudes. You might think that now there would be no further change. But no, the process continues. The amplitude of A continues to decrease and that of B to increase until eventually the displacement of B is equal (or about equal) to that originally given to A, and the displacement of A diminishes toward zero. The starting condition is almost reversed. Now it is easy to predict the sequel. The motion of B is transferred back to A, and so it continues. The energy, originally given to A (and to the spring), does not remain confined to the oscillation of A, but is transferred gradually to B and continues to shuttle back and forth between A and B. Figure 5–3 shows records of actual motions of such a coupled system. The pendulums, whose bobs were dry cells with flashlight bulbs attached, were suspended from the ceiling and were photographed from below by a camera that was pulled steadily along the floor.

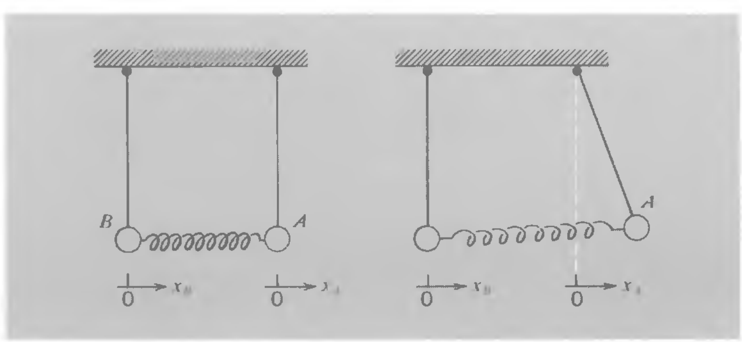


Fig. 5–2 (a) Coupled pendulums in equilibrium position.
(b) Coupled pendulums with one pendulum displaced.

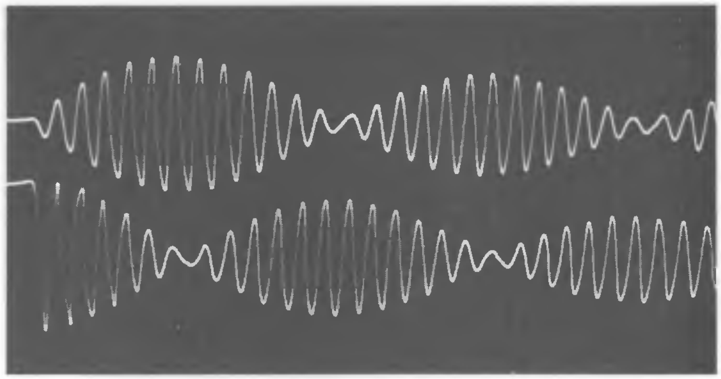


Fig. 5–3 Motion of two identical coupled oscillators (pendulums with flashlight bulbs on the bobs). Pendulum no. 1 was initially at rest at its normal equilibrium position. The damping of the system is quite noticeable. (Photo by Jon Rosenfeld, Education Research Center, M.I.T.)

Of course, it is the coupling spring that is responsible for the observed behavior. As A oscillates, the spring pulls and pushes on B. It provides a driving force that works on B and sets it into motion. At the same time, the spring pulls and pushes on A, sometimes helping, sometimes hindering its motion. But as B begins to move, the action of the spring on A is more to hinder than to help. The net work done on A during one oscillation is negative, and the amplitude of A decreases.

Each of the motions recorded in Fig. 5–3 looks just like a case of beats between two SHM's of the same amplitude but different frequencies. And that is precisely what they are. To account for them in detail is not, however, an obvious matter: Our "feeling" for the physical phenomenon helps us here only qualitatively. But the problem becomes exceedingly simple if we alter the starting conditions somewhat.

SYMMETRY CONSIDERATIONS

Suppose we draw both A and B aside by equal amounts [Fig. 5–4(a)] and then release them. The distance between them equals the relaxed length of the coupling spring and therefore the spring exerts no force on either pendulum. A and B will oscillate in phase and with equal amplitudes, always maintaining the same separation. Each pendulum might just as well be free (uncoupled). Each oscillates with its free natural frequency $\omega_0 (= \sqrt{g/l})$. The equations of motion are

$$\begin{aligned} x_A &= C\cos\omega_0 t \\ x_B &= C\cos\omega_0 t \end{aligned} \qquad (5\text{–}1)$$

where x_A and x_B are the displacements of each pendulum from its equilibrium position. This represents a *normal mode* of the

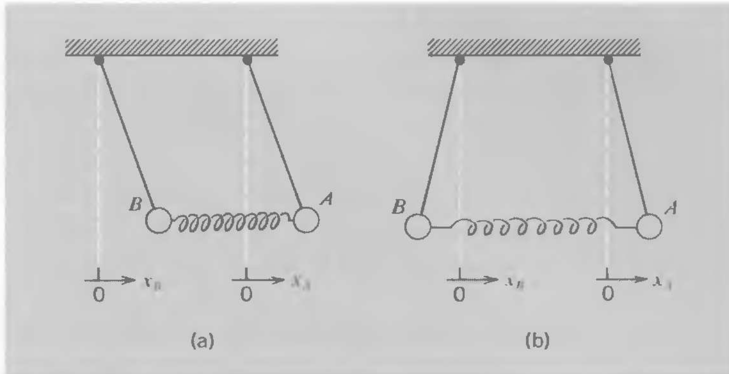


Fig. 5–4 (a) Lower normal mode of two coupled pendulums. (b) Higher normal mode of two coupled pendulums.

coupled system. Both masses vibrate at the same frequency and each has a constant amplitude (the same for both).

How many normal modes can we find? There is only one other. Draw A and B aside by equal amounts but in opposite directions [Fig. 5–4(b)] and then release them. Now, the coupling spring is stretched; a half-cycle later it will be compressed, and it does exert forces. The symmetry of the arrangement tells us that the motions of A and B will be mirror images of each other.

If the pendulums were free and either one were displaced a small distance x, the restoring force would be $m\omega_0^2 x$. But in the present situation the coupling spring is stretched (or compressed) a distance $2x$ and exerts a restoring force of $2kx$, where k is the spring constant. Thus the equation of motion for A is

$$m\frac{d^2x_A}{dt^2} + m\omega_0^2 x_A + 2kx_A = 0$$

or

$$\frac{d^2x_A}{dt^2} + (\omega_0^2 + 2\omega_c^2)x_A = 0$$

where we have let $\omega_c^2 = k/m$. This is an equation for simple harmonic motion of frequency ω' given by

$$\omega' = (\omega_0^2 + 2\omega_c^2)^{1/2} = \left(\frac{g}{l} + \frac{2k}{m}\right)^{1/2}$$

For the given starting conditions, its solution is

$$x_A = D\cos\omega' t \tag{5–2a}$$

The motion of B is the mirror image of A, and therefore

$$x_B = -D\cos\omega' t \tag{5–2b}$$

Each pendulum oscillates with simple harmonic motion, but the action of the coupling spring has been to increase the restoring force and therefore to increase the frequency over that of the uncoupled oscillation. The motions of A and B are clearly always 180° out of phase in this type of oscillation, which constitutes the second normal mode.

It is perhaps worth pointing out that if either of the pendulums is clamped, the angular frequency of the other, under the action of the gravity plus the coupling spring, is equal to $(\omega_0^2 + \omega_c^2)^{1/2}$. Thus if one chooses to regard this motion as being, in a sense, the motion characteristic of one pendulum alone, the normal modes have frequencies that are displaced above or below the single-pendulum value.

THE SUPERPOSITION OF THE NORMAL MODES

In both the above cases, the motion once begun will, in the absence of damping forces, continue without change. No transfer of energy occurs from some one mode of oscillation to another. An important reason for introducing these two easily solved cases is that any motion of the pendulums, in which each starts from rest, can be described as a combination of these two. Let us see how that can be done.

Take an arbitrary moment when pendulum A is at x_A and pendulum B at x_B (Fig. 5–5). The spring is stretched an amount $x_A - x_B$ and therefore pulls on A and B with a force whose magnitude is $k(x_A - x_B)$. Thus the magnitude of the restoring force on A is

$$m\omega_0^2 x_A + k(x_A - x_B)$$

and on B it is

$$m\omega_0^2 x_B - k(x_A - x_B)$$

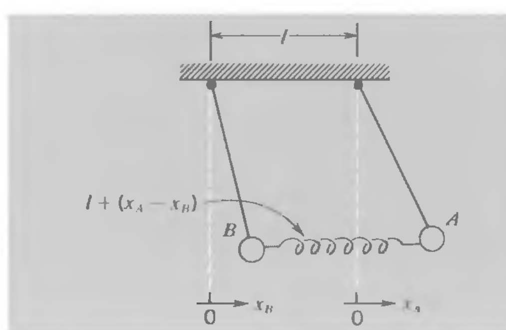


Fig. 5–5 Coupled pendulums in arbitrary configuration.

Therefore, the equations of motion for A and B are

$$m\frac{d^2x_A}{dt^2} + m\omega_0^2 x_A + k(x_A - x_B) = 0$$
$$m\frac{d^2x_B}{dt^2} + m\omega_0^2 x_B - k(x_A - x_B) = 0 \qquad (5\text{–}3)$$

Again letting $\omega_c^2 = k/m$, we can write these as follows:

$$\frac{d^2x_A}{dt^2} + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B = 0$$
$$\frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A = 0 \qquad (5\text{–}4)$$

The first equation, describing the acceleration of A, contains a term in x_B. And the second equation contains a term in x_A. These two differential equations cannot be solved independently but must be solved simultaneously. A motion given to A does not stay confined to A but affects B, and vice versa.

Actually, these equations are not difficult to solve. If we add the two together, we get

$$\frac{d^2}{dt^2}(x_A + x_B) + \omega_0^2(x_A + x_B) = 0$$

and if we subtract the second equation from the first, we get

$$\frac{d^2}{dt^2}(x_A - x_B) + (\omega_0 + 2\omega_c^2)(x_A - x_B) = 0$$

These are familiar equations for simple harmonic oscillations. In the first, the variable is $x_A + x_B$ and the frequency is ω_0. In the second, the variable is $x_A - x_B$ and the frequency is $\omega' = (\omega_0^2 + 2\omega_c^2)^{1/2}$. These two frequencies correspond precisely to those of the two normal modes that we identified previously. If we let $x_A + x_B = q_1$ and $x_A - x_B = q_2$, we have two independent equations in q_1 and q_2:

$$\frac{d^2q_1}{dt^2} + \omega_0^2 q_1 = 0$$
$$\frac{d^2q_2}{dt^2} + \omega'^2 q_2 = 0$$

Possible solutions (although not the most general ones) are

$$\text{(special case)} \qquad q_1 = C\cos\omega_0 t \qquad q_2 = D\cos\omega' t \qquad (5\text{–}5)$$

where C and D are constants which depend upon the initial conditions. [The lack of generality in Eqs. (5–5) can be recognized in the fact that we have set the initial phases equal to zero.]

We have here two independent oscillations. They represent another description of the normal modes, as represented by oscillations of the variables q_1 and q_2 respectively, and these variables are consequently called *normal coordinates.* Changes in the value of q_1 occur independently of q_2 and vice versa.

In terms of our original coordinates, x_A and x_B, the solutions are

$$\text{(special case)}\qquad \begin{aligned} x_A &= \tfrac{1}{2}(q_1 + q_2) = \tfrac{1}{2}C\cos\omega_0 t + \tfrac{1}{2}D\cos\omega' t \\ x_B &= \tfrac{1}{2}(q_1 - q_2) = \tfrac{1}{2}C\cos\omega_0 t - \tfrac{1}{2}D\cos\omega' t \end{aligned} \qquad (5\text{–}6)$$

If $C = 0$, both pendulums oscillate with the frequency ω', or if $D = 0$, with the frequency ω_0. These are the frequencies of the individual normal modes and are called *normal frequencies.* We see that a characteristic of a normal frequency is that both x_A and x_B can oscillate with that frequency.

Let us now apply Eqs. (5–6) to the analysis of the coupled motion shown in Fig. 5–3. The initial conditions (at $t = 0$) are as follows:

$$x_A = A_0 \qquad \frac{dx_A}{dt} = 0 \qquad x_B = 0 \qquad \frac{dx_B}{dt} = 0$$

It may be noted that the conditions on the initial velocities are automatically met by Eqs. (5–6), because differentiation with respect to t gives us terms in $\sin\omega_0 t$ and $\sin\omega' t$ only, all of which go to zero at $t = 0$. From the conditions on the initial displacements themselves we have

$$\begin{aligned} x_A &= A_0 = \tfrac{1}{2}C + \tfrac{1}{2}D \\ x_B &= 0 \;\;\, = \tfrac{1}{2}C - \tfrac{1}{2}D \end{aligned}$$

Therefore,

$$C = A_0 \qquad D = A_0$$

Hence with these particular starting conditions we have, by substitution back into equations (5–6), the following results:

$$\begin{aligned} x_A &= \tfrac{1}{2}A_0(\cos\omega_0 t + \cos\omega' t) \\ x_B &= \tfrac{1}{2}A_0(\cos\omega_0 t - \cos\omega' t) \end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned} x_A &= A_0\cos\left(\frac{\omega' - \omega_0}{2}t\right)\cos\left(\frac{\omega' + \omega_0}{2}t\right) \\ x_B &= A_0\sin\left(\frac{\omega' - \omega_0}{2}t\right)\sin\left(\frac{\omega' + \omega_0}{2}t\right) \end{aligned} \qquad (5\text{–}7)$$

Each of these is a sinusoidal oscillation of angular frequency $(\omega' + \omega_0)/2$, modulated in amplitude in the way discussed in Chapter 2. The amplitude associated with each of the pendulums is zero at the instant when the amplitude associated with the other is a maximum—although the actual *displacement* of the latter at any such instant depends on the instantaneous value of $(\omega' + \omega_0)t/2$.

OTHER EXAMPLES OF COUPLED OSCILLATORS

There are many different ways of coupling two pendulums or other oscillators together; let us consider a few.

In Fig. 5–6 we show how two pendulums may be coupled through an auxiliary mass, $m \ll M$, connected by strings to the major suspending wires. From the symmetry of the arrangement, we can guess that the normal modes will be the motions for which $x_B = \pm x_A$. If $x_A = +x_B = q_1$, the mass m rises and falls with the main masses M, but if $x_A = -x_B = q_2$, the mass m will be highest when the masses M are at their greatest separation, and will fall as the masses approach each other. Thus there are two distinct normal mode frequencies, neither of which (in general) is equal to that of one pendulum alone.

Four other mechanical coupled systems are shown in Fig. 5–7. The first diagram represents two pendulum bobs that are mounted on rigid bars, the upper ends of which are clamped to a wire. The pendulums swing in planes perpendicular to the wire. Unless the pendulums swing in phase, with equal amplitudes, the connecting wire is twisted and provides a coupling torque that is proportional to the difference of angular displacements.

In Fig. 5–7(b) we show another system in which the coupling is provided by elastic restoring forces. Two small masses are mounted at the ends of a hacksaw blade (or other strip of springy metal) which is held at its center by a yielding support. If one

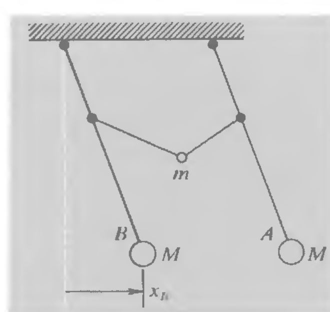


Fig. 5–6 Mass-coupled pendulums.

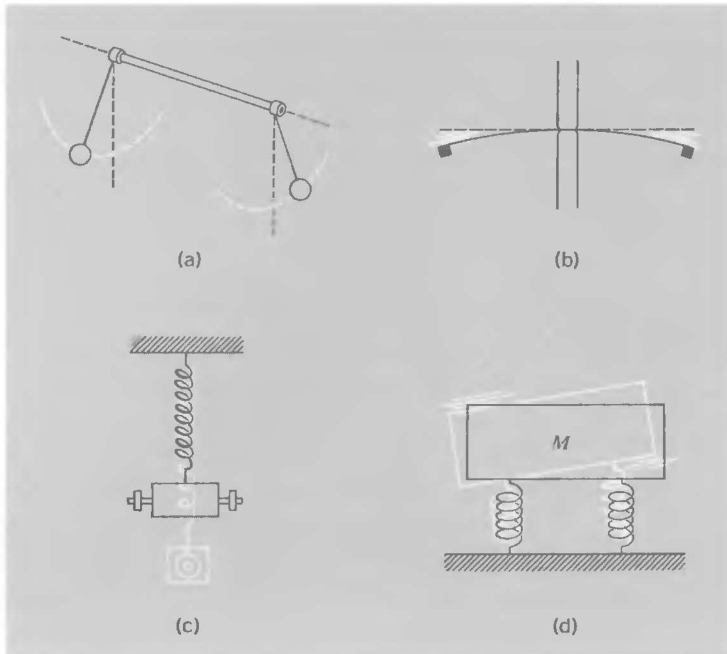


Fig. 5-7 (a) Rigid pendulums coupled by horizontal torsion rod. (b) Masses at ends of metal strip. (c) Wilberforce pendulum. (d) Rectangular block on springs.

mass is pulled aside, as shown, and then released, the motion is quickly transferred to the other mass through a typical superposition of normal modes.

Figure 5-7(c) shows a curious device known as the Wilberforce pendulum.[1] A mass with adjustable outriggers is suspended from a coil spring. If the mass is pulled down and released, the motion is at first a simple vertical oscillation, but as time goes on this oscillation dies down and is replaced by a vigorous rotational oscillation of the mass (about a vertical axis). Then the vertical linear oscillation returns as the rotational oscillation again weakens. It is important for the operation of this toy that the periods of the two types of motion be nearly equal; the adjustable outriggers are there to permit this to be arranged. The coupling between the linear and angular motions comes from the fact that, as we mentioned in Chapter 3, when a coil spring is stretched its end twists a little, or conversely that if it is twisted it tends to lengthen or shorten. By pulling the mass down and twisting it through an appropriate angle, it is possible to release the system so that it oscillates in a normal mode with constant amplitude in both components (linear and angular) of the motion.

[1]Named after L. R. Wilberforce, a British professor of physics, who published a detailed study of it in 1894.

Our last diagram [Fig. 5–7(d)] represents a rectangular block supported on two springs. One mode of this system is a vertical oscillation in which the block remains horizontal and both springs are equally stretched or compressed. But there is another mode in which the springs undergo equal and opposite displacements; the block then performs a twisting oscillation about a horizontal axis, without any change in the height of its center of gravity. A car resting on its front and rear suspensions has some resemblance to this arrangement. If the front end were lifted and then released, one might find the oscillation transferred to the rear at a later time, if damping had not already brought the system to rest.

NORMAL FREQUENCIES: GENERAL ANALYTICAL APPROACH

Suppose it were not easy to discover the normal modes from symmetry considerations, or not easy to solve the simultaneous differential equations. How then could we plough through to a solution? We make use of the characteristic we discussed in connection with Eqs. (5–6). Both x_A and x_B can oscillate with one of the normal frequencies. Let us take, therefore,

$$\begin{aligned} x_A &= C\cos\omega t \\ x_B &= C'\cos\omega t \end{aligned} \qquad (5\text{–}8)$$

and see if there are values of ω and C and C' for which these expressions are solutions of equations (5–4):

$$\begin{aligned} \frac{d^2x_A}{dt^2} + (\omega_0{}^2 + \omega_c{}^2)x_A - \omega_c{}^2x_B &= 0 \\ \frac{d^2x_B}{dt^2} + (\omega_0{}^2 + \omega_c{}^2)x_B - \omega_c{}^2x_A &= 0 \end{aligned} \qquad (5\text{–}4)$$

If there are suitable values of ω, they will then be the normal frequencies. Of course, we have already found that C and C' must be equal in magnitude, but in our present approach to the problem we shall act as though we do not know that yet. Besides, the equality of C and C' is true only in the very special problem we have been considering and is not true in more general cases.

Substituting equations (5–8) into equations (5–4), we get

$$\begin{aligned} (-\omega^2 + \omega_0{}^2 + \omega_c{}^2)C \qquad\qquad - \omega_c{}^2C' &= 0 \\ - \omega_c{}^2C + (-\omega^2 + \omega_0{}^2 + \omega_c{}^2)C' &= 0 \end{aligned}$$

For an arbitrary value of ω, these constitute two simultaneous equations for the unknown amplitudes C and C'. If they are independent equations, there is only one solution—$C = 0, C' = 0$—which simply means that, for an arbitrary value of ω, equations (5–8) are not a solution to the problem.

But if these two equations are not independent—i.e., if the second is just a multiple of the first—then we have in effect only one equation for the two amplitudes C and C'. In this case, C can have any value. But once C is chosen, then C' is fixed.

For what value of ω are the two equations not independent and thus able to yield nonzero solutions for C and C'? From the first equation, we have

$$\frac{C}{C'} = \frac{\omega_c^2}{-\omega^2 + \omega_0^2 + \omega_c^2} \tag{5–9a}$$

and, from the second,

$$\frac{C}{C'} = \frac{-\omega^2 + \omega_0^2 + \omega_c^2}{\omega_c^2} \tag{5–9b}$$

If C and C' are not both zero, the right-hand sides of those equations must be equal. Thus

$$\frac{\omega_c^2}{-\omega^2 + \omega_0^2 + \omega_c^2} = \frac{-\omega^2 + \omega_0^2 + \omega_c^2}{\omega_c^2}$$

or

$$(-\omega^2 + \omega_0^2 + \omega_c^2)^2 = (\omega_c^2)^2$$

Hence

$$-\omega^2 + \omega_0^2 + \omega_c^2 = \pm\omega_c^2$$

$$\omega^2 = \omega_0^2 + \omega_c^2 \pm \omega_c^2$$

We have two solutions for ω; let us call them ω' and ω'':

$$\omega'^2 = \omega_0^2 + 2\omega_c^2$$

$$\omega''^2 = \omega_0^2$$

The positive square roots of these expressions are the two normal frequencies of the system; once again we have arrived at the now familiar results.

We can now get the relation between C and C' for each of the normal modes, from equations (5–9). For $\omega = \omega'$,

$$\frac{C}{C'} = -1$$

and, for $\omega = \omega''$,

$$\frac{C}{C'} = +1$$

Thus we have arrived at two specific forms of equations (5–8) which are solutions to the coupled differential equations of motion [equations (5–4)]:

$$\begin{aligned} x_A &= C\cos\omega_0 t \\ x_B &= C\cos\omega_0 t \end{aligned} \quad \text{and} \quad \begin{aligned} x_A &= D\cos\omega' t \\ x_B &= -D\cos\omega' t \end{aligned} \tag{5–10}$$

Since the magnitude of the amplitude is arbitrary and determined only by the initial conditions, we have used two different symbols (i.e., C and D) to denote the amplitudes associated with the separate normal modes.

The differential equations are linear (only the first powers of x_A, x_B, d^2x_A/dt^2, and d^2x_B/dt^2 appear), and therefore the sum of the two solutions is also a solution:

$$\text{(special case)} \quad \begin{aligned} x_A &= C\cos\omega_0 t + D\cos\omega' t \\ x_B &= C\cos\omega_0 t - D\cos\omega' t \end{aligned} \tag{5–11}$$

Once again we have obtained the solutions previously given by equations (5–6).[1] But this time our approach has been purely analytical and general, with no prior appeal to the symmetry of the system.

Let us complete this discussion by giving the general solution to the equations of free oscillation of this coupled system. It may be readily seen that the differential equations (5–4) are equally well fitted by assuming solutions with nonzero initial phases, although there *is* a systematic phase relationship between x_A and x_B in a particular mode. Specifically, instead of equations (5–10) we may in general have the following:

$$\begin{aligned} &\text{Lower mode:} \quad \begin{aligned} x_A &= C\cos(\omega_0 t + \alpha) \\ x_B &= C\cos(\omega_0 t + \alpha) \end{aligned} \\ &\text{Higher mode:} \quad \begin{aligned} x_A &= D\cos(\omega' t + \beta) \\ x_B &= -D\cos(\omega' t + \beta) \end{aligned} \end{aligned} \tag{5–12}$$

The existence of four adjustable constants then allows us to fit these solutions to arbitrary values of the initial displacements *and* velocities of both pendulums. This removes the restriction

[1]There is a factor of 2 lacking throughout in equations (5–10) as compared with equations (5–6), but this makes no difference at all when one fixes the values of the coefficients via the initial values of x_A and x_B.

to zero initial velocity that required us to label our earlier solutions as special cases.

FORCED VIBRATION AND RESONANCE FOR TWO COUPLED OSCILLATORS

So far we have merely considered the *free* vibrations of a system of two coupled oscillators, thereby discovering the characteristic natural frequencies (just two of them) at which the system is able to vibrate as a kind of unit. But what happens if the system is driven at an arbitrary frequency by an external agency? Our intuition, backed up by actual experience, is that large amplitudes of oscillation occur when the driving frequency is close to one of the natural frequencies, whereas at frequencies far removed from these the response of the driven system is relatively small. We shall consider in detail how this emerges from the equations of motion in the simplest possible case—for two coupled identical pendulums with negligible damping, for which we have already identified the normal modes.

Our discussion will closely parallel the analysis of the forced single oscillator as in Chapter 4. Just as in that case, we shall assume that the damping effects are small enough to be ignored in the equations of motion, but that, nevertheless, perhaps after a very large number of cycles of oscillation, the transient effects have disappeared so that the motion of each pendulum occurs at constant amplitude at the frequency of the driving force.

Let us suppose, then, that a harmonic driving force $F_0 \cos \omega t$ is applied to pendulum A (e.g., by moving its point of suspension back and forth sinusoidally), the motion of pendulum B being controlled only by its own restoring force and the coupling spring. The statement of Newton's law for pendulum B is thus just the same as we had in considering the free vibrations, and the equation for A is modified only to the extent of adding the term $F_0 \cos \omega t$—although this addition represents, of course, a major change in the physical situation. Our two equations of motion thus become the following [see equations (5–3) for the free-vibration equations]:

$$m \frac{d^2 x_A}{dt^2} + m\omega_0^2 x_A + k(x_A - x_B) = F_0 \cos \omega t$$

$$m \frac{d^2 x_B}{dt^2} + m\omega_0^2 x_B - k(x_A - x_B) = 0 \qquad \left(\omega_0^2 = \frac{g}{l}\right)$$

which, dividing through by m, become

$$\frac{d^2x_A}{dt^2} + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B = \frac{F_0}{m}\cos\omega t$$

$$\frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A = 0 \qquad \left(\omega_c^2 = \frac{k}{m}\right)$$

Rather than dealing with x_A and x_B separately, we shall proceed at once to introduce the normal coordinates q_1 ($= x_A + x_B$) and q_2 ($= x_A - x_B$), which, as we have seen, can be used to characterize the motion of the system as a whole. Adding the differential equations above, we get

$$\frac{d^2q_1}{dt^2} + \omega_0^2 q_1 = \frac{F_0}{m}\cos\omega t \tag{5–13a}$$

Subtracting them, we get

$$\frac{d^2q_2}{dt^2} + \omega'^2 q_2 = \frac{F_0}{m}\cos\omega t \tag{5–13b}$$

where

$$\omega'^2 = \omega_0^2 + 2\omega_c^2$$

The simplification of the problem is remarkable. It is just as though we had two harmonic oscillators, of natural frequencies ω_0 and ω'. We can clearly describe the steady-state solutions by the equations

$$\begin{aligned} q_1 &= C\cos\omega t \qquad \text{where } C = \frac{F_0/m}{\omega_0^2 - \omega^2} \\ q_2 &= D\cos\omega t \qquad \text{where } D = \frac{F_0/m}{\omega'^2 - \omega^2} \end{aligned} \tag{5–14}$$

The amplitudes C and D exhibit just the kind of resonance behavior shown for a single oscillator in Fig. 4–1. Having obtained them, we can extract the frequency dependence of the individual amplitudes A and B of the two pendulums, for we have

$$\begin{aligned} x_A &= A\cos\omega t \qquad \text{where } A = \tfrac{1}{2}(C + D) \\ x_B &= B\cos\omega t \qquad \text{where } B = \tfrac{1}{2}(C - D) \end{aligned}$$

These give us the following results:

$$\begin{aligned} A(\omega) &= \frac{F_0}{m}\,\frac{(\omega_0^2 + \omega_c^2) - \omega^2}{(\omega_0^2 - \omega^2)(\omega'^2 - \omega^2)} \\ B(\omega) &= \frac{F_0}{m}\,\frac{\omega_c^2}{(\omega_0^2 - \omega^2)(\omega'^2 - \omega^2)} \end{aligned} \tag{5–15}$$

The variation of these quantities with ω is shown in Fig. 5–8. In the region of frequencies dominated by the lower resonance, the

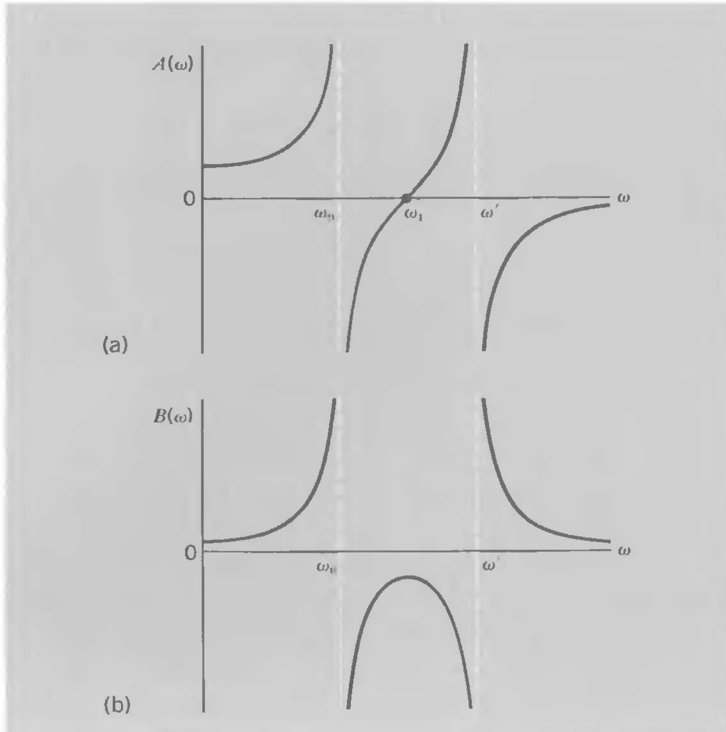


Fig. 5–8 Forced response of two coupled pendulums with negligible damping. The normal modes have the frequencies ω_0 and ω'. (a) Amplitude of first pendulum as a function of driving frequency $[\omega_1 = (\omega_0^2 + \omega'^2)^{1/2}]$. *(b) Amplitude of second pendulum as a function of driving frequency.*

displacements of A and B are always of the same sign—i.e., in phase with one another. In the region of frequencies dominated by the higher resonance, the displacements are of opposite sign and hence 180° out of phase. The introduction of nonzero damping would, as with the single driven oscillator, lead to a smooth variation of phase with frequency as one goes through the resonances.

One feature in particular of Fig. 5–8 might be commented on, because it seems (and is) physically impossible. This is the fact that at a certain frequency ω_1 between the resonances, we have $A = 0$ and B nonzero. Yet from the assumed conditions of the problem it is clear that the periodic forcing of pendulum B depends on the motion of pendulum A. In any real system some small oscillation of the bob of pendulum A would be essential. The frequency ω_1 at which the apparently anomalous situation develops is precisely the natural frequency of a single pendulum, with coupling spring attached, under the circumstance that the other pendulum is held quite fixed—$\omega_1 = (\omega_0^2 + \omega_c^2)^{1/2}$. In the complete absence of damping forces an arbitrarily small driving force of frequency ω_1, caused by arbitrarily small vibrations of pendulum A, would cause an arbitrarily large response in pendulum B. The existence of damping forces, however small, would destroy this condition, and would mean that the amplitude $A(\omega)$, although becoming very small near ω_1, would never fall quite to zero. The full description would now, however, necessitate the detailed consideration of the system as a combination of a pair of oscillators with damping, and the complexity of the analysis would be greatly increased.

The main point to be learned from this analysis is the confirmation that one can trace out the normal modes of a coupled system by means of resonance observations, and that the steady-state motions of the component parts at resonance are just like what they would be for the same system in free vibration at the same frequency.

MANY COUPLED OSCILLATORS

Any real macroscopic body, such as a piece of solid, contains many particles, not just two, so we have the strongest of motives for tackling the problem of an arbitrary number of similar oscillators coupled together. The work of the preceding sections has equipped us to do this. Our investigation of such a system can lead us to a description of the oscillations of a continuous medium, and thence by an easy transition to the analysis of wave motions.

It would be possible for us to go directly from Newton's law to continuum mechanics.[1] But the route we have chosen, via the modes of oscillation of coupled systems, is richer and in essence is more correct—for there is no such thing as a truly continuous medium. Moreover, you may be interested to know that our present route is the one that Newton and his successors themselves took. Perhaps this in itself merits an introductory digression.

Not long after Newton, two members of the remarkable Bernoulli family (John Bernoulli and his son Daniel) embarked on a detailed study of the dynamics of a line of connected masses. They showed that a system of N masses has exactly N independent modes of vibration (for motion in one dimension only). Then in 1753 Daniel Bernoulli enunciated the superposition principle for such a system—stating that the general motion of a vibrating system is describable as a superposition of its normal modes. (You will recall that earlier in this chapter we developed this

[1]As mentioned in the footnote at the beginning of this chapter, you can do this by going directly to Chapter 6.

result for the system of two oscillators.) In the words of Leon Brillouin, who has been a major contributor to the theory of crystal-lattice vibrations[1]:

> This investigation by the Bernoullis may be said to form the beginning of theoretical physics as distinct from mechanics, in the sense that it is the first attempt to formulate laws for the motion of a system of particles rather than for that of a single particle. The principle of superposition is important, as it is a special case of a Fourier series, and in time it was extended to become a statement of Fourier's theorem.

(We shall come to the notions of Fourier analysis in Chapter 6.)

After this preamble let us now turn to the detailed analysis of an N-particle system.

N COUPLED OSCILLATORS

In our treatment of the motion of a two-oscillator system, we confined our attention to oscillations which may be termed longitudinal—the motions of the pendulum bobs have been along the line connecting them. The treatment is quite similar, as we shall soon see, for transverse oscillations where the particles oscillate in a direction perpendicular to the line connecting them. And because transverse oscillations are easier to visualize and to display than longitudinal oscillations, we shall analyze the transverse oscillations of a prototype system of many particles.

Consider a flexible elastic string to which are attached N identical particles, each of mass m, equally spaced a distance l apart. Let us hold the string fixed at two points, one at a distance l to the left of the first particle and the other at a distance l to the right of the Nth particle (Fig. 5–9).

The particles are labeled from 1 to N, or from 0 to $N + 1$ if we include the two fixed ends and treat them as if they were particles with zero displacement. If the initial tension in the string is T and if we confine ourselves to small transverse displacements of the particles, then we can ignore any increase in the tension of the string as the particles oscillate. Suppose, for

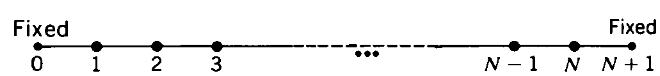


Fig. 5–9 N equidistant particles along a massless string.

[1] L. Brillouin, *Wave Propagation in Periodic Structures*, Dover, New York, 1953.

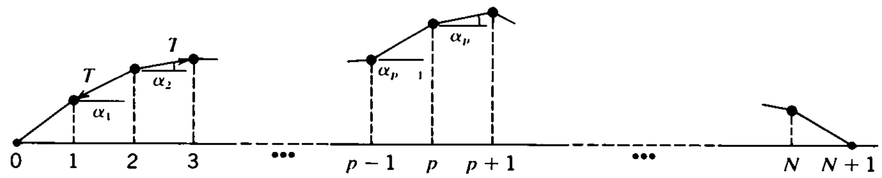


Fig. 5–10 Force diagram for transversely displaced masses on a long string.

example, that particle 1 is displaced to y_1 and particle 2 to y_2 (Fig. 5–10); then the length of string between them becomes $l' = l/\cos\alpha_1$. For $\alpha_1 \ll 1$ rad, then $\cos\alpha_1 \approx 1 - \alpha_1^2/2$ and $l' \approx l(1 + \alpha_1^2/2)$. The increase in length is $l\alpha_1^2/2$, and any increased tension that is proportional to this may be ignored in comparison to any term proportional to the first power of α_1.

In the configuration as shown the resultant x component of force on particle 2 is $-T\cos\alpha_1 + T\cos\alpha_2 = \frac{1}{2}T(\alpha_1^2 - \alpha_2^2)$, a difference between two second-power terms in α. For small values of α_1 and α_2, it is exceedingly small and we shall pay it no attention in what follows.

Figure 5–10 shows a configuration of the particles at some instant of time during their transverse motion. We shall restrict ourselves to y displacements that are small compared to l. The resultant y component of force on a typical particle, say the pth particle, is

$$F_p = -T\sin\alpha_{p-1} + T\sin\alpha_p$$

The approximate values of the sines are

$$\sin\alpha_{p-1} = \frac{y_p - y_{p-1}}{l}$$

$$\sin\alpha_p = \frac{y_{p+1} - y_p}{l}$$

Therefore,

$$F_p = -\frac{T}{l}(y_p - y_{p-1}) + \frac{T}{l}(y_{p+1} - y_p)$$

and this must equal the mass m times the transverse acceleration of the pth particle. Thus

$$\frac{d^2y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0 \qquad (5\text{–}16)$$

where we have put

$$\frac{T}{ml} = \omega_0^2$$

We can write a similar equation for each of the N particles. Thus we have a set of N differential equations, one for each value of p from 1 to N. Remember that $y_0 = 0$ and $y_{N+1} = 0$.

You may find it helpful to consider the simple special cases of Eq. (5–16) for $N = 1$ and $N = 2$. If $N = 1$, we have

$$\frac{d^2y_1}{dt^2} + 2\omega_0{}^2 y_1 = 0$$

There is transverse harmonic motion of angular frequency $\omega_0\sqrt{2} = (2T/ml)^{1/2}$, as one can conclude directly from a consideration of Fig. 5–11(a). If $N = 2$, we have

$$\frac{d^2y_1}{dt^2} + 2\omega_0{}^2 y_1 - \omega_0{}^2 y_2 = 0$$

$$\frac{d^2y_2}{dt^2} + 2\omega_0{}^2 y_2 - \omega_0{}^2 y_1 = 0$$

These are similar to Eqs. (5–4) for the two coupled pendulums, but we now have the simplification that ω_0 and ω_c are equal, so that $\omega_0{}^2 + \omega_c{}^2$ in equations (5–4) corresponds to $2\omega_0{}^2$ here, and $\omega_c{}^2$ there becomes $\omega_0{}^2$ here. The angular frequencies of the normal modes in this case are in a definite numerical relationship; their actual values are ω_0 and $\omega_0\sqrt{3}$. The modes for $N = 2$ are illustrated in Figs. 5–11(b) and (c). The actual configuration of the strings makes almost self-evident the relation between the natural frequencies here, but as we go to larger numbers of particles the results are far less obvious and we must resort to a more general type of analysis.

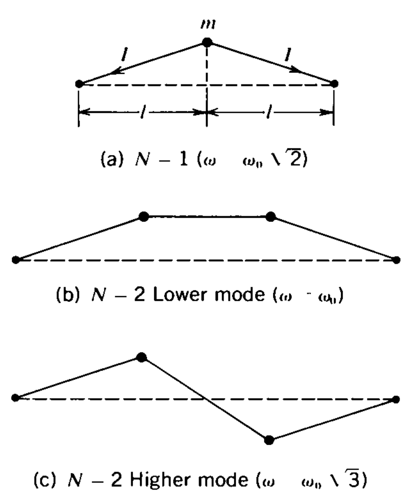


Fig. 5–11 Normal modes of the two simplest loaded-string systems. (a) $N = 1$, one mode only. (b) $N = 2$, lower mode. (c) $N = 2$, higher mode.

FINDING THE NORMAL MODES FOR N COUPLED OSCILLATORS

We apply basically the same analytical technique to our N differential equations as we previously used for the two equations. We seek the normal modes; i.e., we look for sinusoidal solutions such that each particle oscillates with the same frequency. We set

$$y_p = A_p \cos \omega t \qquad (p = 1, 2, \ldots, N) \tag{5–17}$$

where A_p and ω are the amplitude and frequency of vibration of the pth particle. If we can find values of A_p and ω for which equations (5–17) satisfy the N differential equations (5–16), then we have accomplished our purpose. Note that the velocity of any particle can be obtained from equations (5–17) and is

$$\frac{dy_p}{dt} = -\omega A_p \sin \omega t \qquad (p = 1, 2, \ldots, N)$$

Thus, by choosing equations (5–17) as a trial solution, we are automatically restricting ourselves to the additional boundary condition that each particle has zero velocity at $t = 0$; i.e., each particle starts from rest.

Substituting equations (5–17) into the differential equations (5–16), we get

$$\begin{aligned}
(-\omega^2 + 2\omega_0{}^2)A_1 - \omega_0{}^2(A_2 + A_0) &= 0 \\
(-\omega^2 + 2\omega_0{}^2)A_2 - \omega_0{}^2(A_3 + A_1) &= 0 \\
&\vdots \\
(-\omega^2 + 2\omega_0{}^2)A_p - \omega_0{}^2(A_{p+1} + A_{p-1}) &= 0 \\
&\vdots \\
(-\omega^2 + 2\omega_0{}^2)A_N - \omega_0{}^2(A_{N+1} - A_{N-1}) &= 0
\end{aligned}$$

This formidable-looking set of N simultaneous equations can be written more compactly as follows:

$$(-\omega^2 + 2\omega_0{}^2)A_p - \omega_0{}^2(A_{p-1} + A_{p+1}) = 0 \qquad (p = 1, 2, \ldots, N) \tag{5–18}$$

Our earlier boundary condition requiring the ends to be held fixed means that $A_0 = 0$ and $A_{N+1} = 0$.

The question we are asking ourselves is whether all N of these equations can be satisfied by using the same value of ω^2 in each. We saw earlier how to tackle such a problem when only two coupled oscillators were involved. The assumption that a solution existed (other than the trivial one of having all amplitudes equal to zero) led to restrictions on the ratios of the amplitudes [as expressed by equations (5–9)]. We have the same situation in this more complex problem. If we rewrite equations (5–18) as

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} \qquad (p = 1, 2, \ldots, N) \qquad (5\text{–}19)$$

we see that, for any particular value of ω, the right side is constant, and therefore the ratio on the left must be a constant and independent of the value of p. What values can be assigned to the A_p's such that this condition will be satisfied and at the same time give $A_0 = 0$ and $A_{N+1} = 0$?

We shall not pretend to *solve* Eq. (5–19) but will simply draw attention to a remarkable result that gives the key to the problem. Suppose that the amplitude of particle p is expressible in the form

$$A_p = C \sin p\theta \qquad (5\text{–}20)$$

where θ is some angle. If a similar equation is used to define the amplitudes of the adjacent particles $p - 1$ and $p + 1$, we shall have

$$\begin{aligned} A_{p-1} + A_{p+1} &= C[\sin(p - 1)\theta + \sin(p + 1)\theta] \\ &= 2C \sin p\theta \cos \theta \end{aligned}$$

But $C \sin p\theta$ is just A_p, so that we have

$$\frac{A_{p-1} + A_{p+1}}{A_p} = 2 \cos \theta \qquad (5\text{–}21)$$

This means that the recipe represented by Eq. (5–20) is successful. The right-hand side of Eq. (5–21) is a constant, independent of p, which is just what we need so as to have a condition equivalent to Eq. (5–19). It can be used to satisfy all N of the equations (5–18) from which we started. All that remains is to find the value of θ. This we can do by imposing the requirement that $A_p = 0$ for $p = 0$ and $p = N + 1$. The former condition is automatically satisfied; the latter will hold good if $(N + 1)\theta$ is set equal to any integral multiple of π. Thus we put

$$\begin{aligned} (N + 1)\theta &= n\pi \qquad (n = 1, 2, 3, \ldots) \\ \theta &= \frac{n\pi}{N + 1} \end{aligned} \qquad (5\text{–}22)$$

Substituting for θ in Eq. (5–20) we thus get

$$A_p = C \sin\left(\frac{pn\pi}{N + 1}\right) \qquad (5\text{–}23)$$

The permitted frequencies of the normal modes are also determined, for from Eqs. (5–19) through (5–22) we have

$$\frac{A_{p+1} + A_{p-1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = 2\cos\left(\frac{n\pi}{N+1}\right)$$

Therefore,

$$\omega^2 = 2\omega_0^2\left[1 - \cos\left(\frac{n\pi}{N+1}\right)\right]$$
$$= 4\omega_0^2 \sin^2\left[\frac{n\pi}{2(N+1)}\right]$$

Taking the square root of this, we have

$$\omega = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \tag{5–24}$$

PROPERTIES OF THE NORMAL MODES FOR *N* COUPLED OSCILLATORS

Having obtained the mathematical solutions to this problem of N coupled oscillators, let us look more closely at the motions that the equations describe.

First, we observe that, according to Eq. (5–24), different values of the integer n define different normal mode frequencies. It is therefore appropriate to label a mode, and its distinctive frequency, by the value of n. Thus we shall put

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \tag{5–25}$$

Next, we must recognize that the motion of a given particle (or oscillator) depends both on its number along the line (p) and on the mode number (n). The amplitude of its motion can thus be written as follows:

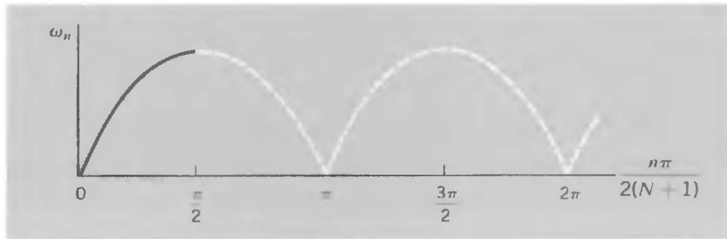
$$A_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right) \tag{5–26}$$

where C_n defines the amplitude with which the particular mode n is excited. The actual displacement of the pth particle when the entire collection of particles is oscillating in the nth mode is thus given by

$$y_{pn}(t) = A_{pn} \cos \omega_n t \tag{5–27}$$

where ω_n and A_{pn} are given by Eqs. (5–25) and (5–26), respectively. The above equation implies that each particle is at rest at the time $t = 0$, but as with the two-oscillator problem we can satisfy arbitrary initial conditions by putting

Fig. 5–12 Graph of the mode frequency as a function of mode number. It is convenient to graph ω_n against the quantity $n\pi/2(N+1)$ rather than against n itself.



$$y_{pn}(t) = A_{pn}\cos(\omega_n t - \delta_n) \tag{5–27a}$$

where each different mode can be assigned its own phase δ_n.

How many normal modes are there? We saw that with two coupled oscillators there were just two normal modes. If your intuition should tell you that with N oscillators there are only N independent modes, you would be right.[1] This fact is, however, somewhat hidden in Eqs. (5–25) and (5–26), because values of ω_n and A_{pn} are defined for every integral value of n. The point is, though, that beyond $n = N$ the equations do not describe any physically new situations.

We can make this clear, as far as the mode frequencies are concerned, with the help of Fig. 5–12. This is a graph of Eq. (5–25) —modified to the extent that ω is defined as being always positive. As we go from $n = 1$ to $n = N$ we find N different characteristic frequencies. At $n = N + 1$, which corresponds to $\pi/2$ on the abscissa, a maximum frequency ω_{max} ($= 2\omega_0$) is reached, but it does not correspond to a possible motion because [as Eq. (5–26) shows] all the amplitudes A_{pn} are zero at this value of n. For $n = N + 2$, we have

$$\begin{aligned}\omega_{N+2} &= 2\omega_0 \sin\left[\frac{(N+2)\pi}{2(N+1)}\right] \\ &= 2\omega_0 \sin\left[\pi - \frac{N\pi}{2(N+1)}\right] \\ &= 2\omega_0 \sin\left[\frac{N\pi}{2(N+1)}\right]\end{aligned}$$

Therefore,

$$\omega_{N+2} = \omega_N$$

Similarly, $\omega_{N+3} = \omega_{N-1}$, and so on. And a similar duplication occurs in every subsequent range of $N + 1$ values of n.

[1]This is for a one-dimensional system. Two dimensions gives $2N$, three dimensions gives $3N$.

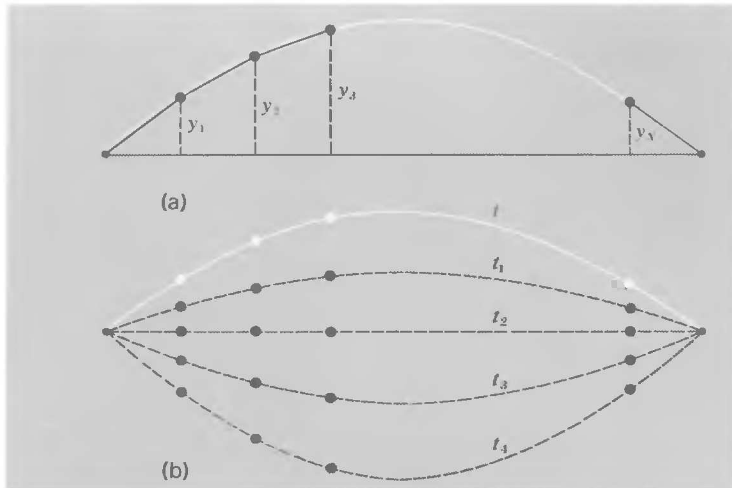


Fig. 5–13 (a) Plot of sin [pπ/(N + 1)] as a function of p. The particles are at the positions defined by integral values of p and are joined by straight segments of string. (b) Positions of particles at various times for lowest mode.

It is only a short step to see that the relative amplitudes of the particles in a normal mode repeat themselves also. Thus, for example, we have, from Eq. (5–26),

$$\begin{aligned} A_{p,N+2} &= C_{N+2}\sin\left[\frac{p(N+2)\pi}{N+1}\right] \\ &= C_{N+2}\sin\left[2p\pi - \frac{pN\pi}{N+1}\right] \\ &= -C_{N+2}\sin\left(\frac{pN\pi}{N+1}\right) \\ &\sim A_{p,N} \end{aligned}$$

and it is easy to show that a similar matching occurs for any other $n > N + 1$.

Let us see what the various normal modes look like. The first mode is given by $n = 1$. The particle displacements are

$$y_{p1} = C_1 \sin\left(\frac{p\pi}{N+1}\right)\cos\omega_1 t \qquad (p = 1, 2, \ldots, N)$$

At a given instant of time, the $C_1 \cos \omega_1 t$ factor is the same for all particles. Only the $\sin[p\pi/(N+1)]$ factor distinguishes the displacements of the different particles. The white curve in Fig. 5–13(a) is a plot of $\sin[p\pi/(N+1)]$ versus p, as p varies continuously from 0 to $N + 1$. Actual particles, however, are located at the discrete values $p = 1, 2, \ldots, N$. The sine curve is therefore only a guide for locating the particles, and the string consists of straight-line segments connecting the particles.

As t increases, each particle oscillates in the y direction with

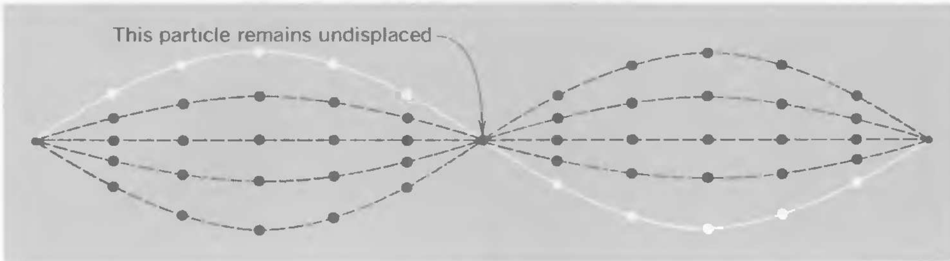


Fig. 5–14 Positions of particles at various times for second mode ($n = 2$).

frequency ω_1. A whole set of sine curves for different values of t, and the corresponding locations of the particles, are shown in Fig. 5–13(b). For the second mode, $n = 2$ and

$$y_{p2} = C_2 \sin\left(\frac{p2\pi}{N+1}\right)\cos\omega_2 t \qquad (p = 1, 2, \ldots, N)$$

The particle displacements at different instants of time are shown in Fig. 5–14. If the number of particles should happen to be odd, there would be one particle at the center of the line and in this mode it would remain at rest, as indicated in Fig. 5–14. Remember that ω_2 differs from ω_1, and therefore this pattern oscillates with a different frequency than the previous one—almost twice as great, in fact.

In Fig. 5–15 we show a set of diagrams of the normal modes for a set of four particles on a stretched string. This displays very beautifully how the pattern of displacements retraces its steps after reaching $n = 5$, even though the sine curves that determine the A_{pn} are all different. These sketches for a small value of N also allow one to appreciate how remarkable it is that the displacements of every particle in every mode for such a system should fall upon a sine curve, when the string connecting them may follow an entirely different path.

LONGITUDINAL OSCILLATIONS

As we explained at the outset, we chose to consider transverse vibrations, rather than longitudinal ones, as a basis for analyzing the behavior of a system comprising a large number of coupled oscillators. The eye and the brain can take in, at a glance, what is happening to each and every particle when a string of masses is set into transverse oscillations. But now let us see how the

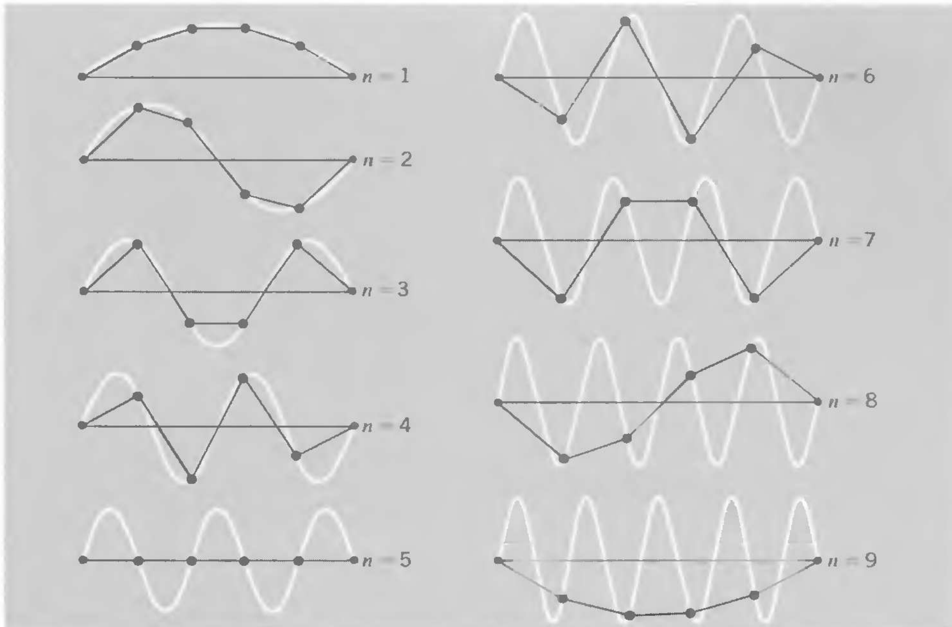


Fig. 5–15 Modes of weighted vibrating string, $N = 4$. Note that $n = 6, 7, 8, 9$ repeat patterns of $n = 4, 3, 2, 1$ with opposite sign. (Adapted from J. C. Slater and N. H. Frank, Mechanics, *McGraw-Hill, New York, 1947.)*

same kind of analysis applies to a system of particles connected by springs along a straight line, and limited to motions along that line. This may seem like a very artificial system, but a line of atoms in a crystal is surprisingly well represented by such a model —and so, to a lesser extent, is a column of gas.

We shall again assume that the particles are of mass m and when at rest are spaced by distances l [Fig. 5–16(a)]. But now the restoring forces are provided by the stretching or compression

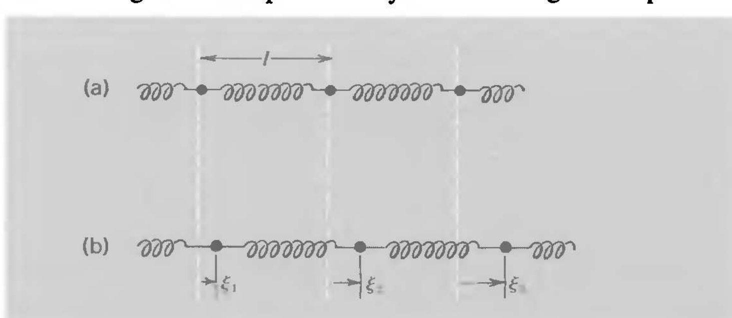


Fig. 5–16 (a) Spring-coupled masses in equilibrium. (b) Spring-coupled masses after small longitudinal displacement.

of the springs; the spring constant for each spring can be written as $m\omega_0^2$. Let the displacements of the masses from their equilibrium positions be denoted by $\xi_1, \xi_2, \ldots, \xi_n$[1] [see Fig. 5–16(b)].

Then the equation of motion of the pth particle is as follows:

$$m\frac{d^2\xi_p}{dt^2} = m\omega_0^2(\xi_{p+1} - \xi_p) - m\omega_0^2(\xi_p - \xi_{p-1})$$

i.e.,

$$\frac{d^2\xi_p}{dt^2} + 2\omega_0^2\xi_p - \omega_0^2(\xi_{p+1} + \xi_{p-1}) = 0 \qquad (5\text{–}28)$$

This has precisely the same form as Eq. (5–16), so we know that mathematically all the features we have discovered for the transverse vibrations of the loaded string have their counterparts in this new system. That is to say, the motion of the pth particle in the nth normal mode is given by

$$\xi_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right)\cos\omega_n t$$

where

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \qquad (5\text{–}29)$$

A very nice quantitative study of such systems has become possible through the use of air suspensions, in which a flow of air (at pressures just a little above atmospheric) from holes in a bearing surface can be made to provide an almost completely frictionless support for objects gliding over the surface. Figure 5–17 shows the results of measurements made with such an apparatus.[2] The masses were each about 0.15 kg, and the spring constants were such that the frequency ω_0 was 5.68 sec^{-1}.

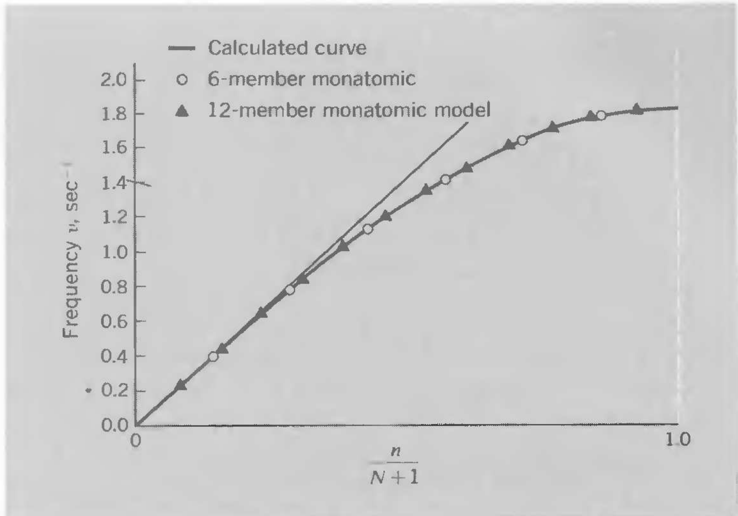
The figure shows the observed frequencies ν_n ($= \omega_n/2\pi$) of the various normal modes, plotted as a function of the variable $n/(N+1)$. The graph contains measurements made with a system of 6 masses (and 7 springs) and with a longer but otherwise similar system of 12 masses (and 13 springs). Since ω_0 was the same for both, the results for the two systems should fall upon the single curve:

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{\omega_0}{\pi}\sin\left(\frac{n}{N+1}\frac{\pi}{2}\right)$$

[1]We use the Greek letter ξ so as to reserve the ordinary x for total distance from one end.

[2]R. B. Runk, J. L. Stull, and O. L. Anderson, *Am. J. Phys.*, **31**, 915 (1963).

Fig. 5-17 Experimental values of mode frequency ν_n plotted against mode number for a line of identical spring-coupled masses. [Note that abscissa is $n/(N+1)$, rather than n; this allows data for two different values of N ($N = 6$ and $N = 12$) to be fitted to same theoretical curve.] [From R. B. Runk, J. L. Stull, and O. L. Anderson, Am. J. Phys., *31, 915 (1963).]*



It may be seen that the experimental values conform extremely well to the theoretical ones.

N VERY LARGE

Suppose now that we allow the number of masses in a coupled system to become very large. To make the discussion explicit, we shall take the case of the transverse vibrations of particles on a stretched string. A real string, just by itself, is in fact already a collection of a large number of closely spaced atoms. Once again we can be sure that our conclusions will apply equally to the line of masses connected by springs in longitudinal vibration.

We shall let N increase but, at the same time, let the spacing l between neighboring particles decrease so that the length of string, $L = (N + 1)l$, remains constant. We shall also decrease the mass of each particle so that the total mass, $M = Nm$, also remains constant.

What happens to the normal frequencies? We have found that

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right]$$

where $\omega_0 = (T/ml)^{1/2}$. First, consider the normal modes for which the mode number n is small. Then as N becomes very large, we can put

$$\sin\left[\frac{n\pi}{2(N+1)}\right] \approx \frac{n\pi}{2(N+1)}$$

Therefore,

$$\omega_n \approx 2\left(\frac{T}{ml}\right)^{1/2} \frac{n\pi}{2(N+1)} = \left(\frac{T}{m/l}\right)^{1/2} \frac{n\pi}{(N+1)l}$$

But $(N+1)l = L$, the total length of the string, and m/l is the mass per unit length (linear density) which we shall denote by μ. Thus, approximately,

$$\omega_n = n\frac{\pi}{L}\left(\frac{T}{\mu}\right)^{1/2} \qquad (n = 1, 2, \ldots) \tag{5-30}$$

In particular,

$$\omega_1 = \frac{\pi}{L}\left(\frac{T}{\mu}\right)^{1/2}$$

and then $\omega_n = n\omega_1$. The normal frequencies are integral multiples of the lowest frequency ω_1. Remember, however, that this *is* only an approximation, even though for $n \ll N$ it is an exceedingly good one.

What about the particle displacements? Previously, we found that, in the nth mode, the displacement of the pth particle is

$$y_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right) \cos \omega_n t$$

Instead of denoting the particle by its p value, we can specify its distance, x, from the fixed end of the string. Now

$$x = pl$$

Hence

$$\frac{pn\pi}{N+1} = \frac{pln\pi}{(N+1)l} = \frac{n\pi x}{L}$$

In place of y_{pn}, we can write $y_n(x, t)$, by which we mean the y displacement at the time t of the particle located at x, when the string is vibrating in the nth mode. Thus

$$y_n(x, t) = C_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t \qquad (n = 1, 2, \ldots) \tag{5-31}$$

As N becomes very large, the x values, which locate the particles, get closer and closer together and x can be taken as a continuous variable going from 0 to L. The white sine curves of Figs. 5–13, 5–14, and 5–15 are now the actual configurations of the string in its different modes. It does not take much imagination to

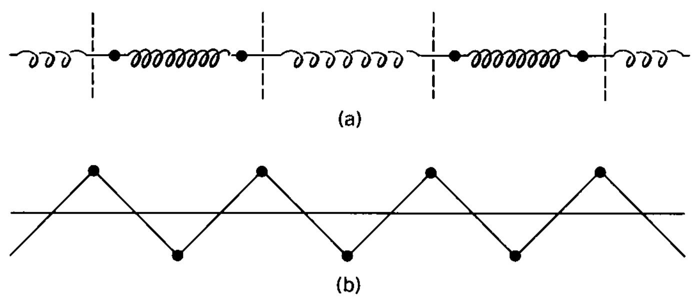


Fig. 5–18 (a) Longitudinal vibrations in the highest mode of a line of spring-coupled masses. (b) Transverse vibrations in the highest mode of a line of masses on a stretched string.

connect such motions with the possibility of wave disturbances traveling along the string, but we shall not proceed to that subject just yet.

Let us now consider the *highest* possible mode, $n = N$. If N is very large, we have

$$\omega_{\text{max}} = 2\omega_0 \sin\left[\frac{N\pi}{2(N+1)}\right] \approx 2\omega_0 \sin\left(\frac{\pi}{2}\right) = 2\omega_0 \qquad (5\text{–}32)$$

In this mode (as we shall show in a moment) each particle has, at every instant, a displacement that is opposite in sign to the displacements of its nearest neighbors, and—except for those particles near to one or the other of the fixed ends—these displacements are almost equal in magnitude. Thus for longitudinal oscillations the situation is somewhat as indicated in Fig. 5–18(a), and for the more readily visualizable case of transverse oscillations it is like Fig. 5–18(b).

This relationship of the adjacent displacements can be inferred with the help of Eq. (5–26):

$$A_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$$

Putting $n = N$, we have

$$A_{p,N} = C_N \sin\left(\frac{pN\pi}{N+1}\right)$$

which we can write as

$$A_{p,N} = C_N \sin(p\pi - \alpha_p)$$

where

$$\alpha_p = \frac{p\pi}{N+1}$$

First, note that in going from p to $p + 1$, the sign of the amplitude is reversed, because the angle $p\pi$ changes from an odd to an even multiple of π (or vice versa) and the angle α_p is less than π for

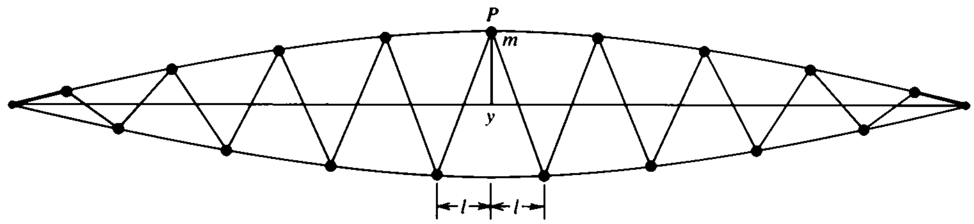


Fig. 5–19 Amplitudes of a complete line of particles in the highest mode for a string fixed at both ends.

every p (since $p \leq N$). This puts successive values of $(p\pi - \alpha_p)$ into opposite quadrants. Thus we can put

$$\text{(highest mode, } n = N) \quad \frac{A_p}{A_{p+1}} = -\frac{\sin\left[\dfrac{p\pi}{N+1}\right]}{\sin\left[\dfrac{(p+1)\pi}{N+1}\right]} \tag{5-33}$$

Notice next that, apart from the alternation of sign, Eq. (5–33) describes a distribution of amplitudes that fit on a half-sine curve drawn between the two fixed ends, as shown in Fig. 5–19 for the case of transverse vibrations of a line of masses.[1] Thus over most of the central region of the line the displacements are almost equal and opposite. Consider, for example, a line of 1000 masses. Then for $100 \leq p \leq 900$ the successive amplitudes differ by less than 1%. It is only toward the ends of the line that the appearance differs markedly from Fig. 5–18(b). It is then easy to see why the frequency should be nearly equal to $2\omega_0$. Consider the particle P in Fig. 5–19. If its displacement at some instant is y, the displacements of its neighbors are both approximately $-y$. Thus if the tension in the connecting strings is T, the transverse component of force due to each is approximately $(2y/l)T$, and the equation of motion of P is given by

$$m\frac{d^2y}{dt^2} \approx -2T\frac{2y}{l}$$

or

$$\frac{d^2y}{dt^2} \approx -\frac{4T}{ml}y = -4\omega_0^2 y$$

(Remember that the magnitudes of the transverse displacements are grossly exaggerated in the diagrams; we really are supposing $y \ll l$, as usual.) The above equation thus defines SHM of angular

[1]Note that this result holds for the highest mode even for small N—see, for example, the fourth diagram in Fig. 5–15.

frequency $2\omega_0$ approximately—and a little further consideration will convince you that the exact frequency is a shade *less* than $2\omega_0$, just as Eq. (5–32) requires.

In all of our discussion of normal modes up until now we have, with good reason, laid great emphasis on the boundary conditions that are applied—whether, for example, the ends of a line of masses are fixed or free. It may, however, have become apparent to you during this last discussion that the properties of the very high modes of a line of very many particles depend relatively little on the precise boundary conditions, even though the low modes are critically dependent on them. Thus the above calculation of the highest mode frequency of the system requires only the realization that the displacements of successive particles are approximately equal and opposite. We should have arrived at the same approximate value of the highest mode frequency if we had assumed that one end of the line was fixed and the other end free. It should be realized, however, that this *is* only approximately true, and that the effect of the precise boundary conditions must always in principle be considered.

NORMAL MODES OF A CRYSTAL LATTICE

We shall not do more than touch on this subject, which, in fact, requires whole books to do it justice. However, the analysis of the previous section carries over in a very successful way to the description of the vibrational modes of solids. This is not too surprising, because, as we have remarked, the interaction between adjacent atoms is, as far as small displacements are concerned, remarkably like that of a spring. And the structure of a solid is a lattice of greater or lesser regularity, justifying the frequently used comparison of a crystal lattice to a three-dimensional bedspring with respect to its vibrational behavior.

If we try to apply Eqs. (5–29) and (5–30) to a solid, we can think of a line of atoms along one of the principal directions in the lattice, so that μ is the total mass of all the atoms per unit length, or the mass of one atom divided by the interatomic separation, l. But what is the tension T? In Chapter 3 we introduced a strong hint for calculating the spring constant due to internal elastic forces. Dimensionally, the ratio T/μ is the same as the ratio Y/ρ of the Young's modulus to the density. The use of this is suggested even more strongly when we think of stretched springs as shown in Fig. 5–16. Thus we shall consider the possibility of describing crystal vibration frequencies $\nu\,(=\omega/2\pi)$ through the following relation:

$$\nu_n = 2\nu_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \quad \text{where } \nu_0 = \frac{1}{2l}\left(\frac{Y}{\rho}\right)^{1/2} \tag{5–34}$$

For solids, as we have seen (see Table 3–1), the values of Y are of the order of 10^{11} N/m^2, so that, because the densities ρ are of the order of 10^4 kg/m^3, the ratio Y/ρ is of the order of 10^7 m^2/sec^2. The interatomic distance l is of the order of 10^{-10} m. Thus we should have

$$\nu_0 \approx 10^{13}\ \text{sec}^{-1}$$

This is the highest frequency that the lattice could support. The low modes are well described by the analogues of Eq. (5–30):

$$\nu_n = \frac{1}{2L}\left(\frac{Y}{\rho}\right)^{1/2}$$

where L is the thickness of the crystal. Thus the *lowest* frequency of vibration of a crystal 1 cm across would be of the order of 10^5 Hz.

To return to the highest possible mode, this is the one in which adjacent atoms are displaced oppositely to one another (see Fig. 5–18). Such motion can be very effectively stimulated by light falling upon an ionic crystal such as sodium chloride, in which the Na^+ and Cl^- ions are always being pushed in opposite directions by the electric field of the light wave. From our very rough calculation, we see that a resonance condition between the light and the lattice might be expected to occur at a frequency of the order of 10^{13} Hz, corresponding to a wavelength of the order of 3×10^{-5} m, or 30μ. This is infrared. Figure 5–20

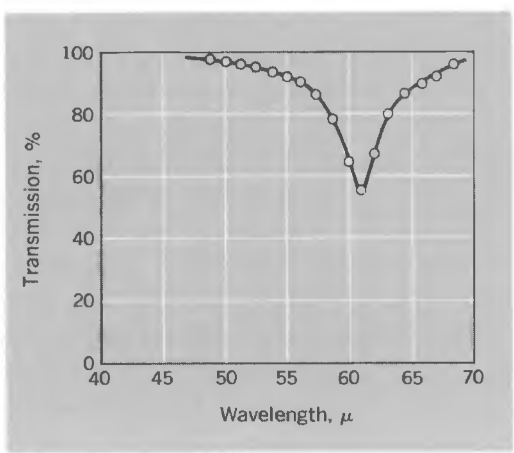
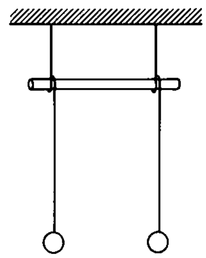


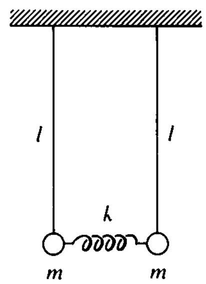
Fig. 5–20 Transmission of infrared radiation through a thin (0.17 μ) sodium chloride film. [*After R. B. Barnes,* Z. Physik, *75*, *723* (*1932*).]

shows a beautiful example of just such a resonance, resulting in increased absorption of light by the crystal at wavelengths in the neighborhood of 60μ. It was observed using an extremely thin slice of NaCl—only about 10^{-7} m thick.

PROBLEMS



5-1 The best way to get a feeling for the behavior of a coupled oscillator system is to make your own, and experiment with it under various conditions. Try making a pair of identical pendulums, connected by a drinking straw that can be set at various distances down the threads (see sketch). Study the motions for oscillations both in the plane of the pendulums (when they move toward or away from one another) and also perpendicular to this plane. Try measuring the normal mode periods and also the period of transfer of motion from one to the other and back. Do your results conform to what the text describes?



5-2 Two identical pendulums are connected by a light coupling spring. Each pendulum has a length of 0.4 m, and they are at a place where $g = 9.8\ \text{m/sec}^2$. With the coupling spring connected, one pendulum is clamped and the period of the other is found to be 1.25 sec exactly.

(a) With neither pendulum clamped, what are the periods of the two normal modes?

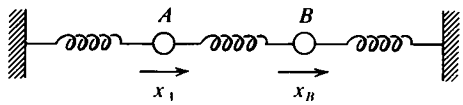
(b) What is the time interval between successive maximum possible amplitudes of one pendulum after one pendulum is drawn aside and released?

5-3 A mass m hangs on a spring of spring constant k. In the position of static equilibrium the length of the spring is l. If the mass is drawn sideways and then released, the ensuing motion will be a combination of (a) pendulum swings and (b) extension and compression of the spring. Without using a lot of mathematics, consider the behavior of this arrangement as a coupled system.

5-4 Two harmonic oscillators A and B, of mass m and spring constants k_A and k_B, respectively, are coupled together by a spring of spring constant k_C. Find the normal frequencies ω' and ω'' and describe the normal modes of oscillation if $k_C{}^2 = k_A k_B$.

5-5 Two identical undamped oscillators, A and B, each of mass m and natural (angular) frequency ω_0, are coupled in such a way that the coupling force exerted on A is $\alpha m(d^2x_B/dt^2)$, and the coupling force exerted on B is $\alpha m(d^2x_A/dt^2)$, where α is a coupling constant of magnitude less than 1. Describe the normal modes of the coupled system and find their frequencies.

5-6 Two equal masses on an effectively frictionless horizontal air track are held between rigid supports by three identical springs, as shown. The displacements from equilibrium along the line of the springs are described by coordinates x_A and x_B, as shown. If either of the masses is clamped, the period $T\,(= 2\pi/\omega)$ for one complete vibration of the other is 3 sec.

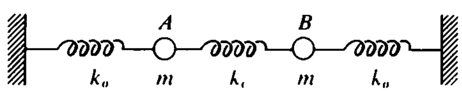


(a) If both masses are free, what are the *periods* of the two normal modes of the system? Sketch graphs of x_A and x_B versus t in each mode. At $t = 0$, mass A is at its normal resting position and mass B is pulled aside a distance of 5 cm. The masses are released from rest at this instant.

(b) Write an equation for the subsequent displacement of each mass as a function of time.

(c) What length of time (in seconds) characterizes the periodic transfer of the motion from B to A and back again? After one cycle, is the situation at $t = 0$ exactly reproduced? Explain.

5-7 Two objects, A and B, each of mass m, are connected by springs as shown. The coupling spring has a spring constant k_c, and the other two springs have spring constant k_0. If B is clamped, A vibrates at a frequency ν_A of 1.81 sec^{-1}. The frequency ν_1 of the lower normal mode is 1.14 sec^{-1}.



(a) Satisfy yourself that the equations of motion of A and B are

$$m\frac{d^2x_A}{dt^2} = -k_0x_A - k_c(x_A - x_B)$$

$$m\frac{d^2x_B}{dt^2} = -k_0x_B - k_c(x_B - x_A)$$

(b) Putting $\omega_0 = \sqrt{k_0/m}$, show that the angular frequencies ω_1 and ω_2 of the normal modes are given by

$$\omega_1 = \omega_0, \qquad \omega_2 = [\omega_0{}^2 + (2k_c/m)]^{1/2},$$

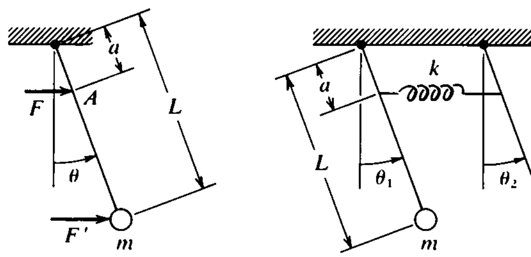
and that the angular frequency of A when B is clamped ($x_B = 0$ always) is given by

$$\omega_A = [\omega_0{}^2 + (k_c/m)]^{1/2}$$

(c) Using the numerical data above, calculate the expected frequency (ν_2) of the higher normal mode. (The observed value was 2.27 sec^{-1}.)

(d) From these same data calculate the ratio k_c/k_0 of the two spring constants.

5-8 (a) A force F is applied at point A of a pendulum as shown. At what angle θ ($\ll$ 1 rad) is the new equilibrium position? What force F', applied at m, would produce the same result?



(a) (b)

Two identical pendulums consisting of equal masses mounted on rigid, weightless rods, are arranged as shown. A light spring (unstretched when both rods are vertical, and placed as shown) provides the coupling.

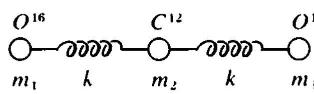
(b) Write down the differential equations of motion for *small-amplitude* oscillations in terms of θ_1 and θ_2. (Neglect damping.)

(c) Describe the motion of the pendulums in each of the normal modes.

(d) Calculate the frequencies of the normal modes of the system.

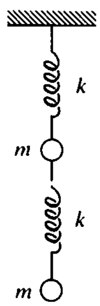
[*Hint:* The symmetry of the system can be exploited to good advantage, particularly in parts (c) and (d), as long as the answers obtained this way are checked in the equations.]

5-9 The CO_2 molecule can be likened to a system made up of a central mass m_2 connected by equal springs of spring constant k to two masses m_1 and m_3 (with $m_3 = m_1$).



(a) Set up and solve the equations for the two normal modes in which the masses oscillate along the line joining their centers. [The equation of motion for m_3 is $m_3(d^2x_3/dt^2) = -k(x_3 - x_2)$ and similar equations can be written for m_1 and m_2.]

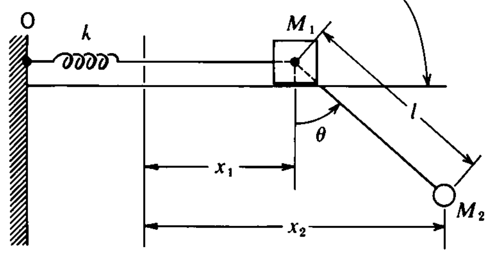
(b) Putting $m_1 = m_3 = 16$ units, $m_2 = 12$ units, what would be the ratio of the frequencies of the two modes, assuming this classical description were applicable?



5-10 Two equal masses are connected as shown with two identical massless springs of spring constant k. Considering only motion in the vertical direction, show that the angular frequencies of the two normal modes are given by $\omega^2 = (3 \pm \sqrt{5})k/2m$ and hence that the ratio of the normal mode frequencies is $(\sqrt{5} + 1)/(\sqrt{5} - 1)$. Find the ratio of amplitudes of the two masses in each separate mode. (*Note:*

You need not consider the gravitational forces acting on the masses, because they are independent of the displacements and hence do not contribute to the restoring forces that cause the oscillations. The gravitational forces merely cause a shift in the equilibrium positions of the masses, and you do not have to find what those shifts are.)

5–11 The sketch shows a mass M_1 on a frictionless plane connected to support O by a spring of stiffness k. Mass M_2 is supported by a string of length l from M_1.



(a) Using the approximation of small oscillations,

$$\sin\theta \approx \tan\theta \approx \frac{x_2 - x_1}{l}$$

and starting from $F = ma$, derive the equations of motion of M_1 and M_2:

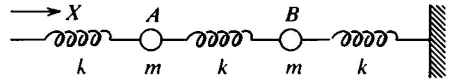
$$M_1\ddot{x}_1 = -kx_1 + M_2\frac{g}{l}(x_2 - x_1)$$

$$M_2\ddot{x}_2 = -\frac{M_2 g}{l}(x_2 - x_1)$$

(b) For $M_1 = M_2 = M$, use the equations to obtain the normal frequencies of the system.

(c) What are the normal-mode motions for $M_1 = M_2 = M$ and $g/l \gg k/M$?

5–12 Two equal masses m are connected to three identical springs (spring constant k) on a frictionless horizontal surface (see figure). One end of the system is fixed; the other is driven back and forth with a displacement $X = X_0 \cos \omega t$. Find and sketch graphs of the resulting displacements of the two masses.

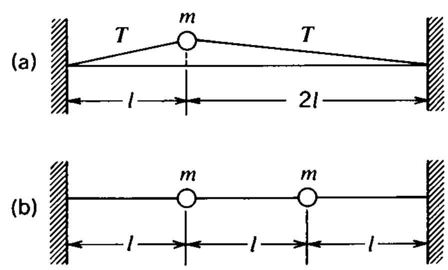


5–13 A string of length $3l$ and negligible mass is attached to two fixed supports at its ends. The tension in the string is T.

(a) A particle of mass m is attached at a distance l from one end of the string, as shown. Set up the equation for small transverse oscillations of m, and find the period.

(b) An additional particle of mass m is connected to the string as shown, dividing it into three equal segments each with tension T. Sketch the appearance of the string and masses in the two separate normal modes of transverse oscillations.

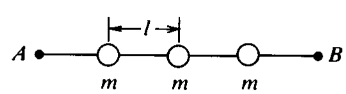
(c) Calculate ω for that normal mode which has the higher frequency.



5-14 To get a feeling for the use of the equation,

$$A_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$$

[Eq. (5-26) in the text], which describes the amplitudes of connected particles in the various normal modes, take the case $N = 3$ and tabulate, in a 3×3 array, the relative numerical values of the amplitudes of the particles ($p = 1, 2, 3$) in each of the normal modes ($n = 1, 2, 3$).



5-15 An elastic string of negligible mass, stretched so as to have a tension T, is attached to fixed points A and B, a distance $4l$ apart, and carries three equally spaced particles of mass m, as shown.

(a) Suppose that the particles have small transverse displacements y_1, y_2, and y_3, respectively, at some instant. Write down the differential equation of motion for each mass.

(b) The appearance of the normal modes can be found by drawing the sine curves that pass through A and B. Sketch such curves so as to find the relative values and signs of A_1, A_2, and A_3 in each of the possible modes of the system.

(c) Putting $y_1 = A_1 \sin \omega t$, $y_2 = A_2 \sin \omega t$, $y_3 = A_3 \sin \omega t$ in the equations (a), use the ratios $A_1:A_2:A_3$ from part (b) to find the angular frequencies of the separate modes.

5–16 Consider a system of N coupled oscillators driven at a frequency $\omega < 2\omega_0$ (i.e., $y_0 = 0$, $y_{N+1} = h \cos \omega t$). Find the resulting amplitudes of the N oscillators. [*Hint:* The differential equations of motion are the same as in the undriven case (only the boundary conditions are different). Hence try $A_p = C \sin \alpha p$, and determine the necessary values of α and C. (*Note:* If $\omega > 2\omega_0$, α is complex and the wave damps exponentially in space.)]

5–17 It is shown in the text that the highest normal-mode frequency of a line of masses can be found by considering a particle near the middle of the line, bordered by particles that have almost equal and opposite displacements to its own. Show that the same frequency can be calculated by considering the *first* particle in the line, acted on by the tension in the segments of string joining it to the fixed end and to particle 2 (see Fig. 5–19 and the related discussion).