PROBLEMS

2.1. A probability density function for a two-dimensional random vector x is defined by

$$f_x(\mathbf{x}) = \begin{cases} Ax_1^2 x_2 & x_1, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the distribution function $F_x(\mathbf{x})$? Use this result to find the numerical value of the constant A.

(b) What is the marginal density $f_{x_2}(x_2)$?

2.2. Let x and y be random variables (one-dimensional random vectors) with density functions

$$f_{x|y}(x|y) = \begin{cases} e^{-(x-y)} & y \leq x < \infty \\ 0 & x < y \end{cases}$$

and

$$f_y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is $f_{xy}(x, y)$? Specify the region where the joint density is nonzero and sketch this region in the xy plane.

(b) What is $f_x(x)$? Don't forget to specify regions of definition.

2.3. The joint density function for the two-dimensional random vectors x and y is

$$f_{xy}(\mathbf{x}, \mathbf{y}) = \begin{cases} x_1 x_2 + 3y_1 y_2 & 0 \leq x_1, x_2, y_1, y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Are x and y statistically independent? Show why or why not.

2.4. The components x_1 and x_2 of a two-dimensional random vector x are statistically independent and have the marginal densities shown in Fig. PR2.4.

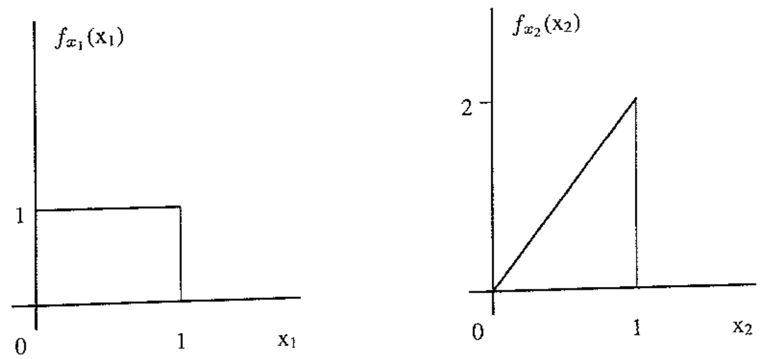

Figure PR2.4

2.21. **(a)** Given the correlation matrix and mean vector

$$\mathbf{R}_x = \begin{bmatrix} 9 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 9 \end{bmatrix} \qquad \mathbf{m}_x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and the transformation

$$y = \begin{bmatrix} 1.0 & 0.5 & 0.5 \\ 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 1.0 \end{bmatrix} x$$

find the mean vector, correlation matrix, and covariance matrix of y.

(b) Repeat this for the transformation

$$y = \begin{bmatrix} 2.0 & 1.0 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} x$$

2.22. Let random variables x and y be defined by

$$x = 3u - 4v$$

$$y = 2u + v$$

where u and v are uncorrelated Gaussian random variables with mean values of 1 and variances of 1.

(a) Find the mean values of x and y.

(b) Find the variances of x and y.

(c) Write an expression for the joint density function of x and y.

(d) Write an expression for the conditional density of y given x.

Hint: What is the mean vector and covariance matrix for the vector whose components are u and v? Use these to find the corresponding quantites for x and y.

2.23. Let v_1, v_2, v_3, v_4 be a set of zero-mean independent random variables with variances equal to $1, 2, 3, 4$. Let x_1, x_2, x_3, x_4 be defined by

$$x_1 = v_1 + v_2 + v_3 + v_4$$

$$x_2 = -v_1 + v_2 + v_3 - v_4$$

$$x_3 = v_1 - v_2 + v_3 - v_4$$

$$x_4 = v_1 + v_2 - v_3 - v_4$$

Show that x_1 and x_2 are uncorrelated, and x_3 and x_4 are uncorrelated, but that x_2 and x_3 are *not* uncorrelated and x_1 and x_4 are *not* uncorrelated.

2.24. Given the correlation matrix

$$\mathbf{R}_x = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and the transformation

$$y = \begin{bmatrix} 1 & -3 \\ 2 & -1 \\ 3 & 2 \end{bmatrix} x$$

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PROBLEMS

4.1. **(a)** Determine the mean and autocorrelation function for the random process

$$x[n] = v[n] + 3v[n-1]$$

where $v[n]$ is a sequence of independent random variables with mean μ and variance σ^2.

(b) Is $x[n]$ stationary?

4.2. Random processes $x[n]$ and $y[n]$ are defined by

$$x[n] = v_1[n] + 3v_2[n-1]$$

and

$$y[n] = v_2[n+1] + 3v_1[n-1]$$

where $v_1[n]$ and $v_2[n]$ are independent white noise processes each with variance equal to 0.5.

(a) What are the autocorrelation functions of x and y? Are these processes wide-sense stationary?

4.22. What are the correlation functions for the real and imaginary parts of the exponential correlation function in Table 4.2? What is the cross-correlation function? Determine also the complex spectral density function for the real and imaginary parts of the process and sketch its poles and zeros.

4.23. A certain real random process is defined by

$$x[n] = A\cos\omega_o n + w[n]$$

where A is a Gaussian random variable with mean zero and variance σ_A^2 and $w[n]$ is a white noise process with variance σ_w^2 independent of A.

(a) What is the correlation function of $x[n]$?

(b) Can the power spectrum of $x[n]$ be defined? If so, what is the power spectral density function?

(c) Repeat parts (a) and (b) in the case when the cosine has an independent random phase uniformly distributed between $-\pi$ and π.

4.24. **(a)** Derive a general expression for the correlation function of the random process defined by (4.103) and show that the process is wide-sense stationary if and only if the amplitudes satisfy the orthogonality condition (4.104).

(b) By decomposing a real sinusoid with real random amplitude and uniform phase into the sum of two complex sinusoids, show that such a random process satisfies the orthogonality condition above and is therefore a stationary random process.

(c) A sampled random square wave with discrete period P has the form

$$x[n] = A\,\mathrm{sqr}(nT - \tau)$$

where A is a real random variable, T is the sampling interval, and τ is a random delay parameter uniformly distributed over $[0, PT]$ and independent of A. This signal is comprised of the fundamental frequency and only odd harmonics. Show that this periodic random process is also stationary.

(d) Give an example of a random process in the form of (4.103) that does *not* satisfy the orthogonality conditions (4.104). Recall that each component of the process is assumed to have uniform phase and amplitude distributed independent of phase.

4.25. A sufficient condition for the random process

$$x[n] = Ae^{\jmath\omega n} = |A|e^{\jmath(\omega n+\phi)}$$

to be wide-sense stationary is that A and ϕ be independent and ϕ be uniformly distributed. This condition also guarrantees that the essential requirement

$$\mathcal{E}\{x[n_1]x[n_0]\} = 0$$

is satisfied for the complex random process. Show that the foregoing condition is only *sufficient* for the stationarity, but not *necessary*. In other words, show by counterexample that $|A|$ and ϕ need not be independent, and that if they are independent, the phase need not be uniformly distributed in order for the random process to be stationary.

4.26. By following a procedure similar to that in Section 4.1.2, prove the positive semidefinite property for the correlation function of a continuous random process. Then by taking the function $a_c(t)$ in Table 4.7 to be an appropriate combination of continuous-time impulses, show that the property (4.130) holds.

(b) What is the correlation function of the output?

5.4. A linear system is defined by

$$y[n] = 0.7y[n-1] + x[n] - x[n-1]$$

(a) Compute the first four values of $R_{yx}[l]$ if it is known that $R_x[l] = \delta[l]$.
(b) What is $R_{xy}[l]$ for $-3 \leq l \leq 3$?
(c) What is the power spectral density function $S_y(e^{\jmath\omega})$?

5.5. A causal linear shift-invariant system is described by the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n-1]$$

Observe that the correlation and cross-correlation functions satisfy the difference equations

$$R_{yx}[l] - \frac{5}{6}R_{yx}[l-1] + \frac{1}{6}R_{yx}[l-2] = R_x[l-1]$$

$$R_y[l] - \frac{5}{6}R_y[l-1] + \frac{1}{6}R_y[l-2] = R_{yx}[1-l]$$

(a) Show that if the input x is a white noise process with unit variance, then the solution to the first equation is

$$R_{yx}[l] = 6\left(\left(\frac{1}{2}\right)^l - \left(\frac{1}{3}\right)^l\right)u[l]$$

where $u[l]$ is the unit step function.

(b) The function above is now used as an input to the second equation. Since the equation is driven with the sum of two exponentials, for $l < 0$ it is reasonable to assume that the response will be of the form

$$R_y[l] = c_1\left(\frac{1}{2}\right)^{-l} + c_2\left(\frac{1}{3}\right)^{-l}; \qquad l < 0$$

Since there is no input for $l > 0$ it is reasonable to assume that the only response will be the transient response, which has a similar form:

$$R_y[l] = c_1'\left(\frac{1}{2}\right)^{l} + c_2'\left(\frac{1}{3}\right)^{l}; \qquad l > 0$$

With these considerations, find the solution to the second equation.

(c) What is the system function $H(z)$ of the original system? Use this to find the z-transform of the correlation function $R_y[l]$.
(d) What is the power spectrum $S_y(e^{\jmath\omega})$?
(e) Find the impulse response of the original system and use the convolution relationships (5.13)–(5.15) to find the output correlation function $R_y[l]$. Check your answer with part (c).

5.6. Find a general closed-form expression for the correlation function of the random process $y[n]$ described by the first-order difference equation

$$y[n] + ay[n-1] = x[n] + bx[n-1]$$

when the input $x[n]$ is white noise with variance σ_o^2.

5.7. A signal with correlation function $R[l] = \left(\frac{1}{2}\right)^{|l|}$ is applied to a linear shift-invariant system with impulse response $h[n] = \delta[n] + \delta[n-1]$.

(a) Compute the correlation function of the output.

(b) Compute the power spectrum of the *input*.

(c) Compute the power spectrum of the *output*.

5.8. A random signal $x[n]$ is passed through a linear system with impulse response

$$h[n] = \delta[n] - 2\delta[n-1]$$

(a) Find the cross-correlation function between input and output $R_{xy}[l]$ if the input is white noise with variance σ_o^2.

(b) Find the correlation function of the output $R_y[l]$.

(c) Find the output power spectral density $S_y(e^{j\omega})$.

5.9. The impulse response of a linear shift-invariant system is given by

$$h[n] = \begin{cases} (-1)^n & 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$$

A white noise process with variance $\sigma_o^2 = 1$ is applied to this system. Call the input to the system $x[n]$ and the output $y[n]$.

(a) What is the cross-correlation function $R_{xy}[l]$? Sketch this function.

(b) What is the correlation function $R_y[l]$ of the output? Sketch this neatly.

5.10. A real random process with correlation function

$$R_x[l] = 2(0.8)^{|l|}$$

is applied to a linear shift-invariant system whose difference equation is

$$y[n] = 0.5y[n-1] + x[n]$$

(a) What is the complex spectral density function of the output $S_y(z)$?

(b) What is the correlation function of the output $R_y[l]$?

5.11. A random process $x[n]$ consists of independent random variables each with uniform density

$$f_x(\mathrm{x}) = \begin{cases} \frac{1}{2} & -1 \le \mathrm{x} \le +1 \\ 0 & \text{otherwise} \end{cases}$$

This process is applied to a linear shift-invariant system with impulse response

$$h[n] = \begin{cases} (\frac{1}{2})^n & n \ge 0 \\ 0 & n < 0 \end{cases}$$

Let the output process be denoted by $y[n]$.

(a) Compute $R_{yx}[l]$.

(b) What is $R_y[l]$?

(c) What is $S_y(z)$? Use this to compute $S_y(e^{j\omega})$.

5.12. A linear shift-invariant system has the impulse response

$$h[n] = 2\delta[n] + \delta[n-1] - \delta[n-2]$$

Thus if $\gamma(\omega)$ is defined as

$$\gamma(\omega) = \ln |H'_{ca}(e^{\jmath\omega})|$$

then it is equivalent to show that

$$\int_{-\pi}^{\pi} |\gamma(\omega)|\, d\omega < \infty$$

(a) Define the function

$$\gamma^+(\omega) = \begin{cases} \gamma(\omega) & \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that

$$\int_{-\pi}^{\pi} e^{2\gamma(\omega)} d\omega > 2\int_{-\pi}^{\pi} \gamma^+(\omega) d\omega$$

Further show that since

$$\int_{-\pi}^{\pi} S_x(e^{\jmath\omega}) d\omega < \infty$$

this implies that

$$\int_{-\pi}^{\pi} \gamma^+(\omega) < \infty$$

(b) Show also that

$$\int_{-\pi}^{\pi} \gamma(\omega) d\omega < \infty$$

[Assume that $H'_{ca}(z)$ has no zeros on the unit circle.]

(c) Finally, show from parts (a) and (b) that

$$2\int_{-\pi}^{\pi} |\gamma(\omega)| d\omega = 2\int_{-\pi}^{\pi} |\ln |H'_{ca}(e^{\jmath\omega})|| d\omega < \infty$$

5.27. Factor the following complex spectral density functions into minimum- and maximum-phase components.

(a)

$$S_x(z) = \frac{-16}{12z - 25 + 12z^{-1}}$$

(b)

$$S_x(z) = -\frac{3 - 10z^{-2} + 3z^{-4}}{3 + 10z^{-2} + 3z^{-4}}$$

5.28. A random process has the complex spectral density function

$$S_x(z) = \frac{z - 2.5 + z^{-1}}{z - 2.05 + z^{-1}}$$

(a) Factor this function into minimum- and maximum-phase terms. What are the poles and zeros?