EE 503 (1)

Review of Linear Algebra:

$\boxed{\text{Linear Space}}$

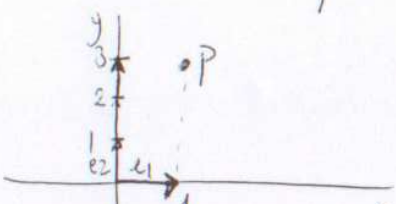
Canonical Basis: For a N Dimensional Space

$$e_1 = [1 \ 0 \ \dots \ 0]$$
$$e_2 = [0 \ 1 \ \dots \ 0]$$
$$\vdots$$
$$e_N = [\underset{1}{0} \ \underset{2}{0} \ \dots \ \underset{N}{1}]$$

The set of e_k is called the canonical basis for $\mathbb{R}^N$.

Any point in $\mathbb{R}^N$ can be expressed trivially with these vectors.

A 2-D example.



$$\underline{P} = \underline{e_1} + 3\underline{e_2}.$$

point P

(2)

To write: $\underline{P} = \underline{e_1} + 3\underline{e_2}$; we need to define vector addition and scalar multiplication for vectors. With the usual definitions for these operations, we end up with a set of points (which are points on the x-y plane) and two "special" operations. The points and operations defined form a linear space.

Linear Space:

$\underline{x}, \underline{y} \in$ Space

1) Linear combination of x, y should be an element of space S. i.e → $z = \alpha \underline{x} + \beta \underline{y}$, $z \in S$. (α, β Scalars)

2) Addition: commutative, associative, has identity and additive inverse.

3) Scalar Multiplication: Distributive over addition, has identity ~~...~~ element.

If all three conditions are satisfied (3)

we have a <u>linear space</u>.

<u>Examples</u>:

a) $\mathbb{R}^N \Longrightarrow$ Linear Space.

b) Functions continuous in $[0,1] \Longrightarrow$ Linear Space.

c) The set of all convergent sequences of real numbers $\Longrightarrow$ Linear Space.

d) The set of all upper triangular matrices $\Longrightarrow$ Linear Space.

<u>Notes</u>: - in (a) physical points that can be measured with a ruler corresponds to the points of linear space

- in (b) functions are the points (elements) of linear space.
- in (c) sequences are points
- in (d) matrices are points.

(4)

<u>Ex</u>: (a) The set of real numbers in the form $(1, a)$ $\Longrightarrow$ Not a linear space.

(b) The set of polynomials of degree n $\Longrightarrow$ Not a Linear Sp.

(c) The set of all vectors of unit length $\Longrightarrow$ Not a Linear Sp.

Why:?

<u>Matrices</u>: Map points to points in a linear space.

a) Space $= \mathbb{R}^2 \longrightarrow \underline{\underline{M}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\underline{\underline{M}}$ maps points in $\mathbb{R}^2$ to $y=x$ line.

b) Space = Polynomials of at most 2nd degree.

$\underline{X} = [x_0 \ \ x_1 \ \ x_2]^T \xrightarrow{\text{corresponds}} x_0 + x_1 z + x_2 z^2$

$\underline{\underline{M}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\underline{\underline{M}}$ maps polynomials to their derivative!

$\underline{\underline{M}}\,\underline{X} = \begin{bmatrix} x_1 \\ 2x_2 \\ 0 \end{bmatrix}$

Change of Basis

Ex: Space: Poly. of 2^{nd} Degree.

$$P(z) = x_0 + x_1 z + x_2 z^2 = x_0 e_0(z) + x_1 e_1(z) + x_2 e_2(z)$$
$$= \left(\frac{x_0+x_1}{2}\right)\psi_0(z) + \left(\frac{x_0-x_1}{2}\right)\psi_1(z) + x_2 \psi_2(z).$$

In here

$e_0(z) = 1$; $e_1(z) = z$; $e_2(z) = z^2$ ← Canonical

$\psi_0(z) = 1+z$; $\psi_1(z) = 1-z$; $\psi_2(z) = z^2$ ← Arbitrary Basis

Conclusion: $P(z)$ is expressed in two different ways:

$$P(z) = 1 + 3z + z^2 \rightarrow \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}_e, \quad \rightarrow \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_\psi.$$

Why we bother for change of basis?

Operators map points to points; but label mapping expressed in different bases ~~map~~ can be very different.

Ex: $P(z) = 1 + 3z + z^2 \rightarrow [1\ 3\ 1]_e^T$, $\rightarrow [2\ -1\ 1]_\psi^T$

$\frac{dP(z)}{dz} = 3 + 2z \rightarrow [3\ 2\ 0]_e^T$, $\rightarrow [\frac{5}{2}\ \frac{1}{2}\ 0]_\psi^T$

d/dt in e

d/d in ψ.

Previously: d/dt in e Basis is found as

$$\underline{\underline{M}}_{d/dz} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

In other words

$$\underline{\underline{M}}^e_{d/dz} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \rightarrow [3 + 2z] \quad \checkmark \text{ok!}$$

$\underline{\underline{M}}_{d/dt}$ operator assigns correct labels!

d/dt in Ψ basis can be found as:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\Psi\text{ Basis: } \Psi_0(z) \arrow[r, dashed, "d/dt \text{ in } \Psi"] \arrow[d, "① \text{ to e basis}"] & \dfrac{\Psi_0(z)+\Psi_1(z)}{2} \\
\text{e. Basis } 1+z \arrow[r, "②", "d/dz"'] & 1 \arrow[u, "③ \text{ to } \Psi \text{ basis}"']
\end{tikzcd}$$

Follow the path ①-②-③ and conclude that $\Psi_0(z)$ is mapped to $\dfrac{\Psi_0(z)+\Psi_1(z)}{2}$ under derivative operator.

Then

$$\underline{M}^{\Psi}_{d/dz} = \begin{bmatrix} 1/2 & ? & ? \\ 1/2 & ? & ? \\ 0 & ? & ? \end{bmatrix}$$

first column vector is written from this information

Since $d/dz\, \Psi_0(z) = \dfrac{\Psi_0(z)+\Psi_1(z)}{2} \longrightarrow \underline{M}^{\Psi}_{d/dt} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$

Other columns can also be found and then

$$M^{\Psi}_{d/dz} = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 1/2 & -1/2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{M}^{\Psi}_{d/dz} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_{\Psi} = \begin{bmatrix} 2.5 \\ 0.5 \\ 0 \end{bmatrix}_{\Psi} \rightarrow (3+2z)$ ok

Conclusion:

Mapping between points is the same; but the mapping of the labels are different!

Sometimes it can be easier to work with one label than the other labels. [Example: Space = Finite energy functions, operation = convolution, label_1 = time domain, label_2 = Fourier domain]

This is called signal representation.

A major application/technique of DSP.

Change of Basis in $\mathbb{R}$:

Alternative basis expressions can simply the mathematical handling.

Ex: $\underline{\underline{T}}_e = \begin{bmatrix} 2 & 0 \\ -6 & 4 \end{bmatrix}$

$\underline{\underline{T}}$: A linear operator in a linear space.

$\underline{\underline{T}}_e$: $\underline{\underline{T}}$ expressed with canonical basis vectors.

$$\underbrace{\underline{x}}_{\text{abstract point}} = x_{e_0}\underline{e}_0 + x_{e_1}\underline{e}_1 = x_{B_0}\underline{B}_0 + x_{B_1}\underline{B}_1$$

$\underline{e}_0, \underline{e}_1 \Rightarrow e_k$ basis; $\underline{B}_0, \underline{B}_1 \Rightarrow B_k$ basis; $x_{B_0}, x_{B_1} \Rightarrow$ expansion coefficients.

$$\underline{x} = \begin{bmatrix} \underline{e}_0 & \underline{e}_1 \end{bmatrix} \begin{bmatrix} x_{e_0} \\ x_{e_1} \end{bmatrix} = \begin{bmatrix} \underline{B}_0 & \underline{B}_1 \end{bmatrix} \begin{bmatrix} x_{B_0} \\ x_{B_1} \end{bmatrix}$$

Mapping of expansion coef.

$$\begin{bmatrix} x_{B_0} \\ x_{B_1} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{B}_0 & \underline{B}_1 \end{bmatrix}^{-1} \begin{bmatrix} \underline{e}_0 & \underline{e}_1 \end{bmatrix}}_{} \begin{bmatrix} x_{e_0} \\ x_{e_1} \end{bmatrix}$$

$$\begin{bmatrix} x_{B_0} \\ x_{B_1} \end{bmatrix} = \underline{\underline{M}}_{e\to B} \begin{bmatrix} x_{e_0} \\ x_{e_1} \end{bmatrix}$$

$\underline{\underline{M}}_{e\to B}$: Change of basis matrix (expansion coefficients are transferred to the other basis)

Next Question:

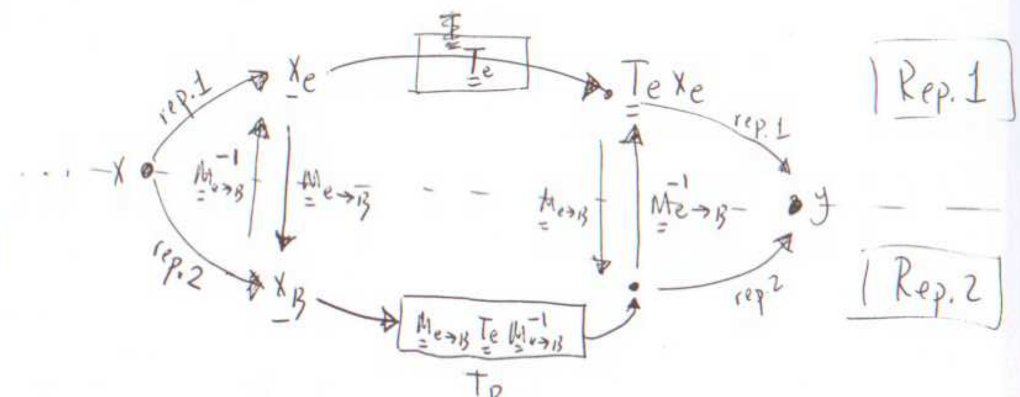
* What is the equivalent of $\underline{\underline{T}}_e$ in B basis?

$$\underline{y}_e = \underline{\underline{T}}_e \underline{x}_e \longrightarrow \underline{\underline{M}}_{e\to B}\underline{y}_e = \underline{\underline{M}}_{e\to B}\underline{\underline{T}}_e \underline{x}_e$$

$$\underline{y}_B = \underline{\underline{M}}_{e\to B}\underline{\underline{T}}_e \underline{\underline{M}}_{B\to e}\underline{x}_B$$

$$\underline{y}_B = \underbrace{\left(\underline{\underline{M}}_{e\to B}\underline{\underline{T}}_e \underline{\underline{M}}_{e\to B}^{-1}\right)}_{\underline{\underline{T}}_B}\underline{x}_B$$

$$\underline{\underline{T}}_B = \underline{\underline{M}}_{e\to B}\underline{\underline{T}}_e \underline{\underline{M}}_{e\to B}^{-1}$$



Linear Mapping $\underline{\underline{T}}$ expressed in two ways.

The simplest possible $\underline{\underline{T}}_B$ operator is the diagonal matrix. A diagonal $\underline{\underline{T}}_B$ indicates
that expansion coefficients in $\{B\}$ basis are decoupled, that is k^{th} expansion coefficient only effects the k^{th} expansion ~~#~~ coefficient of the output.

$$\underbrace{\begin{bmatrix} y_{B_1} \\ y_{B_2} \\ \vdots \\ y_{B_N} \end{bmatrix}}_{\text{Coef's at output.}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix} \underbrace{\begin{bmatrix} x_{B_1} \\ \vdots \\ x_{B_N} \end{bmatrix}}_{\text{Coef.'s at input}}$$

How to diagonalize $\underline{\underline{T}}e$?

$$\underline{\underline{T}}e = \begin{bmatrix} 2 & 0 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}_{\underline{\underline{T}}_B} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^{-1}$$

$$\underline{\underline{T}}e \begin{bmatrix} 1 & \boxed{0} \\ 3 & \boxed{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \boxed{4} \end{bmatrix}$$

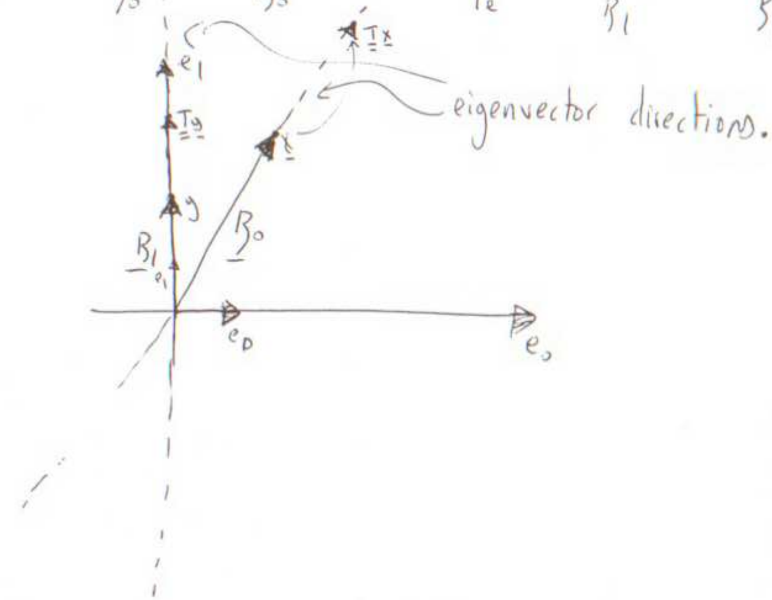
B_0 → eigenvector of $\underline{\underline{T}}e$ for eigenvalue λ_0

B_1 → eigenvector of $\underline{\underline{T}}e$ for eigenvalue λ_1

$\underline{\underline{T}}e\, \underline{B}_0 = \lambda_0 \underline{B}_0 \quad | \quad \underline{\underline{T}}e\, \underline{B}_1 = \lambda_1 \underline{B}_1$

That is if we form the basis B from the eigenvectors of matrix $\underline{\underline{T}}e \xrightarrow{\text{then}} \underline{\underline{T}}_B$ becomes a diagonal matrix.

$$(*) \quad \underbrace{\begin{bmatrix} 2 & 0 \\ -6 & 4 \end{bmatrix}}_{\underline{\underline{T}}e} \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{B_0} = 2 \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{B_0} \; ; \; \underbrace{\begin{bmatrix} 2 & 0 \\ -6 & 4 \end{bmatrix}}_{Te} \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{B_1} = 4 \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{B_1}.$$



Any point on the eigenvector lines lies on the same line after the application $\underline{\underline{T}}e$ operator. (See the equation (*)). The after mapping points are the scaled versions of input points. Any point $\underline{x}$ that can be expressed as the linear combination of eigenvectors can be trivially mapped to $\underline{\underline{T}}\,\underline{x}$ point by scaling the expansion coefficients of $\underline{x}$.

Similar Matrices:

$$\hat{\underline{\underline{T}}} = \underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\underline{M}}^{-1}$$

, the matrices $\underline{\underline{T}}$ and $\hat{\underline{\underline{T}}}$ satisfies the relation on the left side are called similar matrices.

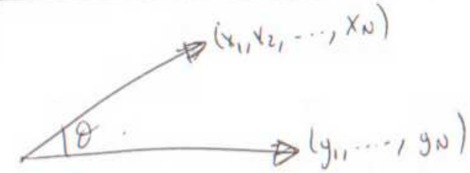
Similar matrices ~~are~~ can be interpretted as the expression of the mapping $\underline{\underline{T}}$ in alternative coordinate axes.

Facts:
a) eigenvalues of $\{\underline{\underline{T}}\}$ = eigenvalues of $\{\hat{\underline{\underline{T}}}\}$.
b) rank $\underline{\underline{T}}$ = rank $\hat{\underline{\underline{T}}}$
c) det $\underline{\underline{T}}$ = det $\hat{\underline{\underline{T}}}$
d) $\underline{\underline{T}}^T$ is similar to $(\hat{\underline{\underline{T}}})^T$
e) $\underline{\underline{T}}^k$ is similar to $(\hat{\underline{\underline{T}}})^k$

(most of these facts follow from fact a))

Orthonormal Bases:

Angle Between Vectors:



$\underline{x}$ and $\underline{y}$ are N dim vectors.

The angle θ can be expressed as.

$$\cos\theta = \frac{\sum_{k=1}^{N} x_k y_k}{\sqrt{\left(\sum_{k=1}^{N} x_k^2\right)\left(\sum_{k=1}^{N} y_k^2\right)}} \Rightarrow$$

The numerator can be expressed as. $\underline{x}^T\underline{y}$ (or $\underline{y}^T\underline{x}$) where $\underline{x}$ and $\underline{y}$ are assumed to be column vectors as before, and the denominator is $\sqrt{(\underline{x}^T\underline{x})\,(\underline{y}^T\underline{y})}$

$$\cos\theta = \frac{\underline{x}^T\underline{y}}{\|\underline{x}\|\,\|\underline{y}\|}$$

where

$$\|x\| = \sqrt{\underline{x}^T\underline{x}} \quad ; \|y\| = \sqrt{\underline{y}^T\underline{y}}$$

1) The canonical coordinate axis are mutually orthogonal.

$$\underline{e}_k^T \underline{e}_\ell = 0 \qquad \text{when } k \neq \ell$$

2) The angle between two vectors before and after mapping with a linear operator $\underline{\underline{M}}$:

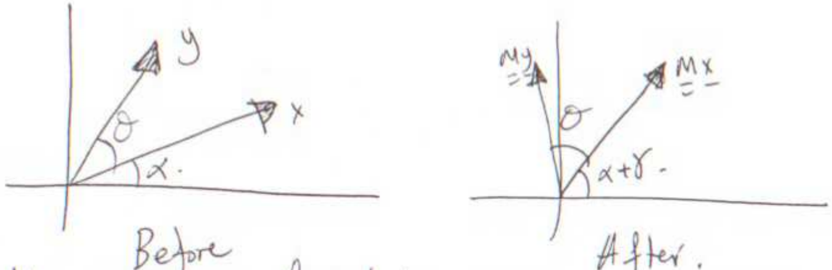
$$\cos\vartheta_b = \frac{\underline{x}^T \underline{y}}{\sqrt{\underline{x}^T\underline{x} \;\; \underline{y}^T\underline{y}}}$$

$$\cos\vartheta_{after} = \frac{(\underline{\underline{M}}\underline{x})^T(\underline{\underline{M}}\underline{y})}{\sqrt{(\underline{\underline{M}}\underline{x})^T(\underline{\underline{M}}\underline{x})\;(\underline{\underline{M}}\underline{y})^T(\underline{\underline{M}}\underline{y})}}$$

$$= \frac{\underline{x}^T \underline{\underline{M}}^T \underline{\underline{M}}\, \underline{y}}{\sqrt{(\underline{x}^T \underline{\underline{M}}^T \underline{\underline{M}}\, \underline{x})(\underline{y}^T \underline{\underline{M}}^T \underline{\underline{M}}\, \underline{y})}}$$

If $\underline{\underline{M}}^T \underline{\underline{M}} = \underline{\underline{I}} \leftarrow$ identity matrix then.

$$\cos\vartheta_{before} = \cos\vartheta_{after}.$$



The after picture is formed by rotating the vectors by an angle of γ. Therefore before and after mapping angles stay the same.

The matrices that does not alter the angle between the vectors are very important for Signal processing. These matrices can be used as change of basis matrices. Orthogonal change of basis matrices map orthogonal bases to orthogonal bases.

That is,

$\{\underline{e}_k\}$ be an orthogonal basis $\longrightarrow$ $\underline{e}_k^T \underline{e}_\ell = 0 \quad k \neq \ell.$

$\downarrow$

$\{\underline{\underline{M}}\underline{e}_k\}$ is also an orthogonal basis since $\underline{e}_k^T \underbrace{\underline{\underline{M}}^T \underline{\underline{M}}}_{\underline{\underline{I}}} \underline{e}_\ell = 0 \quad k \neq \ell$

Change of Basis with Orthogonal Representations:

$$\underline{\underline{M}}_{e\to B} = [\underline{B}_0 \ \ \underline{B}_1]^{-1} [\underline{e}_0 \ \ e_1] \qquad (\text{for a 2-D space})$$

where $\{\underline{B}_0, \underline{B}_1\}$ are basis vectors for B ~~space~~ representation

$\{\underline{e}_0 \ \ \underline{e}_1\}$ are basis vectors for e ~~space~~ representation

Assume that $\{\underline{e}\}$ is the canonical coordinate system.

$\{\underline{B}\}$ is a set of orthogonal vectors.

Then $[\underline{e}_0 \ \ \underline{e}_1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\underbrace{[\underline{B}_0 \ \ \underline{B}_1]}_{\text{Orthogonal matrix}}^{-1} = \begin{bmatrix} \underline{B}_0^T \\ \underline{B}_1^T \end{bmatrix}$

Then point x in $\{e\}$ basis $(\underline{x}_e)$ is mapped to the coordinate set in $\{B\}$ basis with the relation:

$$\underline{X}_B = \underline{\underline{M}}_{e\to B} \ \underline{x}_e$$

$$\underline{x}_B = \begin{bmatrix} \underline{B}_0^T \cdots \\ \underline{B}_1^T \cdots \end{bmatrix} \underline{x}_e$$

Since $\underline{X}_B = \begin{bmatrix} x_{B_0} \\ x_{B_1} \end{bmatrix} \rightarrow X = x_{B_0} \underline{B}_0 + x_{B_1} \underline{B}_1 \xrightarrow{\text{then}}$

$$\rightarrow X = x_{B_0} \cdot \underline{B}_0 + x_{B_1} \underline{B}_1$$

$$\underline{X} = (\underline{B}_0^T \underline{x}_e) \underline{B}_0 + (\underline{B}_1^T \underline{x}_e) \underline{B}_1$$

$\underline{B}_0^T \underline{x}_e$ is a row vector times a column vector $\rightarrow$

$\rightarrow$ a scalar. This scalar represents the angle between $\underline{B}_0$ and $\underline{x}_e$ (when properly interpretted, i.e. $\cos^{-1}$ argument).

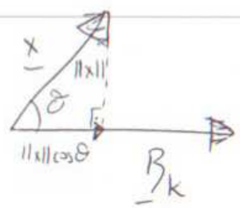
The same operation can be repeated in N-Dim spaces:

$$\underline{X} = \sum_{k=0}^{N-1} a_k \underline{B}_k \qquad \text{where} \quad a_k = \underline{B}_k^T \underline{x}_e$$

A more general notation for $\underline{B}_k^T \underline{x}_e$ is

$$(\underline{x}_e, \underline{B}_k)$$

which is called inner product. As before inner product is related to the angle between vectors. And its full definition can be given in terms of projections.



$$\cos\vartheta = \frac{\left(\underline{B}_k^T \underline{x}\right)}{\|\underline{x}\|\,\|\underline{B}_k\|}$$

$$\left(\underline{B}_k^T \underline{x}\right)\underline{B}_k = \Big(\|x\|\,\underbrace{\|\underline{B}_k\|}_{1}\cos\vartheta\Big)\,\underline{B}_k.$$

$$= \left(\|x\|\cos\vartheta\right)\underline{B}_k$$

The last relation states that $\underline{x}$ is mapped to a vector along with the basis vector B_k with length $\|x\|\cos\vartheta$. This operation is called the projection of $\underline{x}$ over $\underline{B}_k$.

then

$$\underline{X} = \sum_{k=0}^{N-1} (\underline{x}, B_k)\,\underline{B}_k.$$

$\underline{X}$ is formed by sum of projections over the basis vectors of $\{\underline{B}_k\}$ rep.