Derivation of length extension formulas for complementary sets of sequences using orthogonal filterbanks

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A method for the construction of complementary sets of sequences using polyphase representation of orthogonal filterbanks is presented. It is shown that the case of two-channel filterbanks unifies individually derived length extension formulas for complementary sequences into a common framework and the general M-channel case produces novel formulas for the extension of complementary sets of sequences. The presented technique can also be used to generate polyphase and multi-level sequences.

Introduction: Binary complementary sequences derived by Golay [1] which have been extended to polyphase sequences in [2] and complementary sets of sequences in [3] have recently found many applications in modern communications such as peak-to-average power ratio control, multi-carrier communication systems, least square MIMO channel identification [4–7]. Complementary sequences are defined as a pair of vectors the a-periodic autocorrelation of which sum to a delta function. More explicitly for the complementary pair $(\mathbf{x}, \mathbf{y})$, the addition of autocorrelation functions result in $r_x(k)+r_y(k)=2N\delta(k)$ where $r_x(k)=\sum_{n=0}^{N-k-1} x(n)x(n+k)$ is a-periodic autocorrelation sequence for a length N vector.

Complementary sequences are originally designed for the binary alphabet of $\{1,-1\}$. Golay has defined and showed that binary complementary sequences exist for lengths $N=2^{\alpha}10^{\beta}26^{\gamma}$ and given length extension formulas to generate length $2N$ sequences from length N sequences [1]. The extension formulas are significant for communication applications which require a multitude of complementary sequences as in [5, 6]. It is shown that the extension formulas can generate $m!2^m$ complementary pairs of length 2^m, which is more than sufficient for many applications.

In [8], it has been shown that complementary sequences and even-shift orthogonal sequences can be put into one-to-one correspondence, i.e. it is possible to generate a unique complementary sequence from an even-shift sequence and vice versa. The definition of even-shift orthogonal sequences is as follows: $r_s(2k)=0$, $r_c(2k)=0$ for $k\neq 0$ where $(\mathbf{s}, \mathbf{c})$ are even-shift orthogonal vectors and $r_s(k)$, $r_c(k)$ are the autocorrelation functions as defined before. Furthermore for even-shift orthogonal vectors $r_{sc}(2k)=0$ for all k, i.e. the cross-correlation $r_{sc}(k)=\sum_{n=0}^{N-k-1} s(n)c(n+k)$ is identical to zero for all even samples.

The two-channel orthogonal filterbanks which have been extensively studied in signal processing literature [9, 10] are closely related to even-shift sequences. The perfect reconstruction and alias cancellation requirements for two-channel filterbanks can be written as $r_s(2k)=\delta(k)$, $r_c(2k)=\delta(k)$, $r_{sc}(2k)=0$ where $s[n]$ and $c[n]$ are the impulse response of the analysis filters in the bank [9, p. 148]. The conditions for the perfect construction is identical to defining properties of even-shift orthogonal sequences with the exception that even-shift orthogonal sequences are only defined for binary alphabet, while the analysis filters are defined in complex field. Recently, the link between even-shift orthogonal sequences and orthogonal filterbanks has been used to define anti-podal paraunitary matrices which have been studied in the noise amplification problem of channel equalisers for OFDM systems, [11]. In this Letter, we use the link between complementary sequences and orthogonal filterbanks to first unify a number of different approaches on complementary sequence extension and then generalise these methods to the complementary sets of sequences.

Polyphase representation: The condition for perfect reconstruction condition can be alternatively expressed in polyphase matrix form as $\mathbf{H}_p(z)\mathbf{H}_p^T(z^{-1})=\mathbf{I}$ [9, p. 150]. $\mathbf{H}_p(z)$ is called the polyphase matrix representation of the filterbank. For the two-channel case, it is composed of even-odd indexed entries of lowpass and highpass analysis filters $c(z)$, $d(z)$:

$$\mathbf{H}_p(z)=\begin{bmatrix} c_{even}(z) & c_{odd}(z) \\ d_{even}(z) & d_{odd}(z) \end{bmatrix} \tag{1}$$

The polyphase matrices satisfying the perfect reconstruction condition are called paraunitary matrices. The factorisation theorem of paraunitary matrices by Vaidyanathan [10] states that an arbitrary paraunitary filterbank can be factorised into a product of orthogonal and diagonal matrices with delay elements:

$$\mathbf{H}_p(z)=\mathbf{R}(\Theta_m)\Delta(z)\mathbf{R}(\Theta_{m-1})\Delta(z)\cdots\mathbf{R}(\Theta_1)\Delta(z)\mathbf{R}(\Theta_0) \tag{2}$$

In the expression above $\mathbf{R}(\Theta)$ is an orthogonal matrix and $\Delta(z)$ is a diagonal matrix with diagonal entries $[1, z^{-1}, z^{-2}, \ldots z^{N-1}]$. We use presented relations between complementary sequences, even-shift sequences and filterbanks to unify a number of known results on the extension of complementary sequences.

Extension formulas for complementary sequences: In the literature, various extensions formulas for Golay sequences have been proposed. If $\{x, y\}$ denote a pair of complementary sequences of length N, then the following is true:

(i) Concatenation of complementary sequences in the form $\{[\mathbf{xy}], [\mathbf{x}(-\mathbf{y})]\}$ results in a length $2N$ complementary sequence [1]. Since the complementary sequences are identical to even-shift orthogonal, we can use Vaidyanathan's theorem to express the length extension formula in terms of polyphase matrices:

$$\mathbf{H}_{2N}(z)=\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\sqrt{2}\mathbf{R}(\pi/4)}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & z^{-N/2} \end{bmatrix}}_{\Delta(z^{-N/2})}\underbrace{\begin{bmatrix} x_{even}(z) & x_{odd}(z) \\ y_{even}(z) & y_{odd}(z) \end{bmatrix}}_{\mathbf{H}_N(z)} \tag{3}$$

The matrices $\mathbf{H}_N(z)$ and $\mathbf{H}_{2N}(z)$ represent the original and extended complementary sequences in polyphase format. The matrix $\mathbf{R}(\pi/4)$ represents a 45° rotation matrix and $\Delta(z)$ is the diagonal matrix with delay elements. The extended complementary sequences can be converted from polyphase format to vector format with $\mathbf{H}_{2N}(z^2)[1\ z^{-1}]^T$ operation.

(ii) The second formula which is also due to Golay [1], interleaves two complementary sequences in the form

$$\{[x_0y_0x_1y_1\ldots x_{N-1}y_{N-1}], [x_0(-y_0)x_1(-y_1)\ldots x_{N-1}(-y_{N-1})]\} \tag{4}$$

to generate length $2N$ complementary sequences. This result can also be put in polyphase form $\mathbf{H}_{2N}(z)=\sqrt{2}\mathbf{R}(\pi/4)\Delta(z^{-1})\mathbf{H}_N(z^2)$ where all matrices are defined as in the previous extension formula.

(iii) In [2], Sivaswamy extended the results of Golay to non-binary but polyphase (unit norm) alphabets. The extension formula for polyphase sequences is $\{[\mathbf{x}(W\mathbf{y})], [\mathbf{x}(-W\mathbf{y})]\}$ where W is an arbitrary unit norm complex number. This extension can be related to filterbanks as $\mathbf{H}_{2N}(z)=\sqrt{2}\mathbf{R}(\pi/4)\Delta([W, -W])\Delta(z^{-N/2})\mathbf{H}_N(z^2)$ where $\Delta([W, -W])$ is the diagonal matrix with $[W, -W]$ on the diagonal.

(iv) In [12], the formulas of Sivaswamy have been further extended to multi-level sets. It can be shown that the extension is equivalent to $\mathbf{H}_{2N}(z)=\sqrt{2}\mathbf{R}(\pi/4)\Delta([W, -W])\Delta(z^{-M})\mathbf{H}_N(z^2)$, where M is an arbitrary integer.

Complementary sets of sequences: Complementary sets of sequences have been generalised from complementary sequences in [3, 13]. The defining relation for the complementary sets of sequences is $\sum_{n=1}^{T} r_x n(k)=NT\delta(k)$, where we have T sequences $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_T}\}$ of length N the autocorrelation of which sum to a delta function. Complementary sets of sequences have recently attracted significant attention in multi-user and MIMO communications [4–7]. To our knowledge, there are no results in the literature on the length extension for complementary sets of sequences except the result of Darnell and Kemp [14] reporting a special of case of the general framework that we present. In this Letter, we show that the extension formulas for the complementary sets can be directly generalised from the formulas of complementary sequences by utilising M-channel filterbanks instead of two channels in the polyphase representation. For example, the concatenation method of Golay described previously can be generalised to sets as follows:

$$\mathbf{H}_{4N}(z)=\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}}_{R(\Theta)}\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-N/4} & 0 & 0 \\ 0 & 0 & z^{-N/2} & 0 \\ 0 & 0 & 0 & z^{-3N/4} \end{bmatrix}}_{\Delta(z^{-N/4})} \times \underbrace{\begin{bmatrix} x_0(z) & x_1(z) & x_2(z) & x_3(z) \\ y_0(z) & y_1(z) & y_2(z) & y_3(z) \\ z_0(z) & z_1(z) & z_2(z) & z_3(z) \\ w_0(z) & w_1(z) & w_2(z) & w_3(z) \end{bmatrix}}_{\mathbf{H}_N(z)} \tag{5}$$

This results in the extension for the complementary sequences of length $N\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ to another complementary set of sequences of length $4N$ as follows:

$$\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\} \rightarrow \left\{ \begin{array}{c} [\mathbf{xyzw}], \\ [\mathbf{x}(-\mathbf{y})\mathbf{z}(-\mathbf{w})], \\ [\mathbf{xy}(-\mathbf{z})(-\mathbf{w})], \\ [\mathbf{x}(-\mathbf{y})(-\mathbf{z})\mathbf{w}] \end{array} \right\} \tag{6}$$

Similarly other extension formulas can also be generalised by replacing 2×2 rotation matrices in the expansion with $N \times N$ orthogonal matrices. In the expansion above we have used 4×4 Hadamard matrix for this purpose.

Conclusions: Using the connection between complementary sequences and orthogonal filterbanks, we have generalised the extension formulas for complementary sequences to complementary sets. The generalisation allows us to generate polyphase, multi-level complementary sets and brings the known extension formulas for complementary sets into the same framework. We finally note that the extension formulas mentioned in this Letter generate N-shift-orthogonal sequences which are subsets of complementary sets of sequences. For the even-shift case, it has been shown that even-shift orthogonal sequences completely characterise complementary sequences [8]. Such a result is not valid N-shift orthogonal sequences. The main application of the presented formulas is in multi-user, multi-carrier or multi-input multi-output communication applications.

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