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A fine-resolution frequency estimator using an arbitrary number of DFT coefficients

Umut Orguner *, Çağatay Candan

Department of Electrical and Electronics Engineering, Middle East Technical University (METU), 06800 Ankara, Turkey

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ABSTRACT

A method for the frequency estimation of complex exponential signals observed under additive white Gaussian noise is presented. Unlike competing methods based on relatively few Discrete Fourier Transform (DFT) samples, the presented technique can generate a frequency estimate by fusing the information from all DFT samples. The estimator is shown to follow the Cramer–Rao bound with a smaller signal-to-noise ratio (SNR) gap than the competing estimators at high SNR.

1. Introduction

Frequency estimation of complex exponential signals is a fundamental problem of signal processing frequently arising in diverse applications such as speech processing, communications, radar systems, and analog-to-digital converter specifications. Recently, some computationally efficient methods having a minor gap from the theoretical limits have been proposed [1–3]. In this letter, a novel estimator with a further reduced gap is described.

The conventional method for the frequency estimation, which is the maximum likelihood (ML) estimator for the data collected under additive white Gaussian noise (AWGN), is the periodogram. The frequency estimate via periodogram is calculated by finding the peak location in the magnitude Discrete-Time Fourier Transform (DTFT) spectrum, [4, p. 543]. Typically, the frequency estimate, i.e., the maxima location, is calculated in two-stages. In the first stage, the DTFT spectrum is calculated over a uniformly spaced grid with the Discrete Fourier Transform (DFT) to get a coarse frequency estimate. In the next stage, a local search is conducted in the vicinity of the coarse estimate to refine the peak location.

Frequency estimation methods can be categorized as non-iterative and iterative methods. Non-iterative methods apply a simple formula on a few DFT outputs obtained in the first stage and generate the final frequency estimate with an almost negligible additional computational cost [3,5–7]. In principle, these methods do not conduct a local search around the peak detected in the first stage; but refine the estimate by reprocessing the samples in the DFT spectrum. The iterative methods, on the other hand, carry out iterations treating the refined estimate of an earlier iteration as the coarse estimate of the next until convergence [8,9]. It should be noted that the iterative methods may require as few as two iterations for an excellent performance.

In this work, a novel fine-frequency estimator, which can be used iteratively, is presented. Different from some other estimators in the literature, the suggested estimator is not limited to a few samples in the DFT spectrum; but can utilize all available samples. It has been shown that the utilization of more samples in the spectrum leads to a reduced gap between the estimator performance and the Cramer–Rao lower bound.

2. Preliminaries

We consider a complex exponential signal of unknown amplitude, phase and frequency which is corrupted by

* Corresponding author.
E-mail addresses: umut@metu.edu.tr (U. Orguner), ccandan@metu.edu.tr (Ç. Candan).

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additive white Gaussian noise:

$$r[n] = Ae^{j(2\pi fn + \phi)} + w[n], \quad n = 0, \dots, N-1. \tag{1}$$

The frequency variable f in (1) is the normalized frequency defined over the interval $[0, 1)$. As in [3] and earlier works, the frequency in this paper is denoted in terms of the DFT bins, that is $f = (k_p + \delta)/N$ where k_p is an integer in $[0, N-1]$ and δ is a real number in $-1/2 < \delta < 1/2$. The goal of fine-frequency estimation is the estimation of δ. The white noise $w[\cdot]$ is circularly symmetric complex-valued Gaussian distributed with zero mean and σ_w^2 variance, $w[n] \sim \mathcal{CN}(0, \sigma_w^2)$. The signal-to-noise ratio (SNR) definition adopted in this work is the input SNR, $\text{SNR} = A^2/\sigma_w^2$.

The first stage of the proposed method estimates the coarse part of the frequency (k_p) from the N-point DFT of $r[\cdot]$ as described in the introduction section. In the second stage, the fractional part of the frequency (δ) is estimated. The Cramer–Rao lower bound (CRB) for the estimation of δ (or equivalently f) in terms of DFT bins can be written from [10] as

$$\text{CR} - \text{Bound} = \frac{6}{(2\pi)^2 N \text{SNR}} \quad \text{for } N \gg 1. \tag{2}$$

The estimator given in [3] uses three DFT bins to construct its estimate and is shown to have an SNR gap between 2.2 dB (for $\delta = 0$) and 4.5 dB (for $|\delta| = 0.5$) from the CRB. A major contribution of the present paper is the reduction of the mentioned performance gap via the utilization of additional DFT bins in the spectrum.

3. Proposed estimator

The DFT of the signal $r[\cdot]$ in (1) in the absence of noise can be written as follows:

$$R[k_p + k] = Ae^{j\phi} e^{j\pi \frac{N-1}{N}(\delta - k)} \frac{\sin(\pi(\delta - k))}{\sin(\pi(\delta - k)/N)}. \tag{3}$$

Expanding the sine functions appearing on the numerator and denominator of the relation on the right hand side of (3), the ratio of the bins $R[k_p + k]$ and $R[k_p]$ can be written as

$$\gamma_k \triangleq \frac{R[k_p + k]}{R[k_p]} = \frac{e^{j(\pi/N)k} \sin(\pi\delta/N)}{\sin(\pi\delta/N)\cos(\pi k/N) - \sin(\pi k/N)\cos(\pi\delta/N)}. \tag{4}$$

The expression (4) is the key relation upon which the estimator of δ is based. We can obtain the following two equations for: γ_k and γ_{-k} from (4).

$$\gamma_k e^{-j(\pi/N)k}\left[\sin\left(\frac{\pi}{N}\delta\right)\cos\left(\frac{\pi}{N}k\right) - \sin\left(\frac{\pi}{N}k\right)\cos\left(\frac{\pi}{N}\delta\right)\right] = \sin\left(\frac{\pi}{N}\delta\right),$$

$$\gamma_{-k} e^{j(\pi/N)k}\left[\sin\left(\frac{\pi}{N}\delta\right)\cos\left(\frac{\pi}{N}k\right) + \sin\left(\frac{\pi}{N}k\right)\cos\left(\frac{\pi}{N}\delta\right)\right] = \sin\left(\frac{\pi}{N}\delta\right). \tag{5}$$

Summing the two expressions in (5), dividing both sides by $\cos((\pi/N)\delta)\cos((\pi/N)k)$ and rearranging, we get

$$\tan\left(\frac{\pi}{N}\delta\right) = \tan\left(\frac{\pi}{N}k\right)\frac{\gamma_k e^{-j(\pi/N)k} - \gamma_{-k} e^{j(\pi/N)k}}{\gamma_k e^{-j(\pi/N)k} + \gamma_{-k} e^{j(\pi/N)k} - \frac{2}{\cos(\pi k/N)}}$$

$$= \tan\left(\frac{\pi}{N}k\right)\text{Re}\left\{\frac{R[k_p+k]e^{-j(\pi/N)k} - R[k_p-k]e^{j(\pi/N)k}}{R[k_p+k]e^{-j(\pi/N)k} + R[k_p-k]e^{j(\pi/N)k} - \frac{2R[k_p]}{\cos\left(\frac{\pi}{N}k\right)}}\right\} \tag{6}$$

where we used the definitions of γ_k, γ_{-k} with the fact that both sides of the first equation are real to obtain the second equation.

Solving for δ then gives

$$\widehat{\delta}_k = \frac{N}{\pi}\tan^{-1}\left(\tan\left(\frac{\pi}{N}k\right) \times \text{Re}\left\{\frac{R[k_p+k]e^{-j(\pi/N)k} - R[k_p-k]e^{j(\pi/N)k}}{R[k_p+k]e^{-j(\pi/N)k} + R[k_p-k]e^{j(\pi/N)k} - \frac{2R[k_p]}{\cos\left(\frac{\pi}{N}k\right)}}\right\}\right). \tag{7}$$

It should be noted that the estimator (7) uses three DFT bins, that is, $R[k_p]$ and two bins which are k bins away from k_p, to form the estimate $\widehat{\delta}_k$. In the next section, a way of fusing several $\widehat{\delta}_k$ estimates for different k values ($0 < k < N/2$) is described in order to harvest the total energy in all DFT samples and to reduce the estimation error.

3.1. Fusion of frequency estimates

Unlike other estimators such as [1,3], it is possible to generate a multitude of frequency estimates by substituting $k = \{1, 2, \dots, N/2-1\}$ into (7). Since the estimates for large values of k, say $k > 4$, are derived from the DFT samples at a relatively large distance from the main lobe, the mean square error (MSE) of these estimates can be large in comparison to the estimates with small k values. Therefore, intuitively, the estimates with large k values should have less effect on the fused estimate than those with small k values. In the present section, we describe a method which has this property for fusing the estimates $\widehat{\delta}_k$ for $k = \{1, 2, \dots, N/2-1\}$.

Fusion rule: Under high SNR and small $|\delta|$ assumptions, the estimates $\widehat{\delta}_k$, $k = \{1, \dots, N/2-1\}$ are approximately uncorrelated. (See the companion document [11] for a derivation.) Hence a possible fusion approach to generate an estimate for δ given $\widehat{\delta}_k$ for $k = \{1, 2, \dots, N/2-1\}$ is the weighted average of $\widehat{\delta}_k$ which is called as the best linear unbiased estimate (BLUE). The weights for the BLUE are proportional to the reciprocal of MSE, i.e., 1/MSE, for each $\widehat{\delta}_k$ [4, p. 139]. Hence, the fusion of the estimates $\widehat{\delta}_k$ is achieved with the relation given below.

$$\widehat{\delta}_F = \frac{\sum_{k=1}^{N/2-1} \frac{1}{(\text{MSE})_k}\widehat{\delta}_k}{\sum_{k=1}^{N/2-1} \frac{1}{(\text{MSE})_k}} \tag{8}$$

where $(\text{MSE})_k$ refers to the MSE of the estimator $\widehat{\delta}_k$.

Weights of the fusion rule: Under high SNR and small $|\delta|$ assumptions, MSE of the estimator $\widehat{\delta}_k$ can be approximated as (See the companion document [11] for a derivation.)

$$(\text{MSE})_k \approx \frac{1}{4N\text{SNR}} \frac{\sin^2(\pi k/N)}{\pi^2/N^2} \frac{N^2 \sin^2(\pi\delta/N)}{\sin^2(\pi\delta)}. \tag{9}$$

Table 1
Iterated application of the fine-frequency estimators.

0	Given $r[n]$ for $n=\{0,1,\dots,N-1\}$
1	Set $\hat{\hat{\delta}} \leftarrow 0$
2	Evaluate N-point DFT on $r[\cdot]$ and find k_p
3	Generate $\hat{\delta}$ via (7), (10) or (13)
4	Set $\hat{\hat{\delta}} \leftarrow \hat{\hat{\delta}} + \hat{\delta}$
5	If $\hat{\delta}$ is sufficiently small,
6	Go to Step 11
7	Else,
8	Set $r[n] \leftarrow r[n] \exp(-j2\pi\hat{\delta}n/N)$, $n=\{0,1,\dots,N-1\}$
9	Go to Step 2
10	End of If
11	The frequency estimate is $k_p+\hat{\hat{\delta}}$ in bins or $2\pi/N(k_p+\hat{\hat{\delta}})$ radians/sample

Using the approximation (9) in the BLUE fusion formula (8) and making the required cancelations by noting that $1/(\mathrm{MSE})_k \propto 1/\sin^2(\pi k/N)$, we obtain

$$\hat{\delta}_F = \frac{\sum_{k=1}^{N/2-1} \frac{1}{\sin^2(\pi k/N)} \hat{\delta}_k}{\sum_{k=1}^{N/2-1} \frac{1}{\sin^2(\pi k/N)}} \tag{10}$$

which is independent of the true δ-value. Hence, the BLUE formula (10) provides us with a realizable way of fusing the estimates $\hat{\delta}_k$, $k=\{1,2,\dots,N/2-1\}$. An exact analytical relation can be written for the normalization constant appearing in the denominator of (10) as [12]

$$\sum_{k=1}^{N/2-1} \frac{1}{\sin^2(\pi k/N)} = \sum_{k=1}^{N/2-1} \csc^2(\pi k/N) = \frac{1}{6}(N+2)(N-2). \tag{11}$$

Considering the fact that the MSE of the BLUE in (8) can be calculated as $(\mathrm{MSE})_F = (\sum_{k=1}^{N/2-1} 1/(\mathrm{MSE})_k)^{-1}$ [4, p. 139], the MSE of $\hat{\delta}_F$ in (10) is approximately given as

$$(\mathrm{MSE})_F \approx \frac{6N}{(2\pi)^2 \mathrm{SNR}(N-2)(N+2)} \frac{N^2 \sin^2(\pi\delta/N)}{\sin^2(\pi\delta)}, \tag{12}$$

where we used (9) and (11), which converges to the CRB in (2) for large N and sufficiently small $|\delta|$ values.

Number of DFT bins: It is possible to use only the first $K<N/2-1$ DFT bins to form an estimate at the expense of MSE performance as below.

$$\hat{\delta}_F^K = \frac{\sum_{k=1}^{K} \frac{1}{\sin^2(\pi k/N)} \hat{\delta}_k}{\sum_{k=1}^{K} \frac{1}{\sin^2(\pi k/N)}}. \tag{13}$$

An approximate empirical formula for the number of DFT bins to use for the estimator $\hat{\delta}_F^K$ in (13) to work with at most $1/\alpha$ times (where $\alpha \in (0,1)$ is a scalar) the MSE of $\hat{\delta}_F$ (i.e., $(\mathrm{MSE})_F$) in (12)) is given as (See [11] for a detailed analysis.)

$$K = \min\left(N/2-1, \mathrm{round}\left(\frac{0.607927}{1-\alpha}\right)\right). \tag{14}$$

Iterated application: It can be noted from (12) that the estimator performance increases as $|\delta|$ gets smaller. The estimators having such a property can be utilized in an iterative fashion to improve the performance for all δ-values. This is achieved by changing, in effect, the operating point of the estimator from large $|\delta|$ to small $|\delta|$ as shown in Table 1. Since the proposed estimator $\hat{\delta}_F$ and the Aboutanios and Mulgrew estimator [9] can reach relatively close MSE values to CRB for small $|\delta|$, their iterations can boost the performance significantly. However, some other estimators, like Candan's estimator [3] and the ones described in [1], do not equally benefit from the iterative operation; since the SNR gaps of these estimators from the CRB for small $|\delta|$ values are relatively large.

4. Numerical experiments

Three sets of numerical results are given to illustrate the performance of the suggested method. We consider the signal $r[\cdot]$ in (1) with $N=16$ and $f=(2+\delta)/16$, i.e., $k_p=2$. The proposed estimators $\hat{\delta}_k$ and $\hat{\delta}_F$ along with the estimators of Candan [3] and Aboutanios and Mulgrew [9] are evaluated on 100,000 Monte Carlo (MC) runs for different values of δ. In each MC-run different noise $w[\cdot]$ and phase ($\phi \sim \mathrm{Uniform}[0,2\pi]$) realizations are used. The root-mean-square errors (RMSE) of the algorithms calculated over the MC-runs are illustrated with magnifications in the figures when necessary.

Experiment #1: Fig. 1(a) and (b) shows the RMSE performances of the estimators $\hat{\delta}_k$ and $\hat{\delta}_F$ (with only a single iteration) for different values of k for the true δ-values 0.1 and 0.4 respectively. Note that the RMSE performance significantly degrades for high values of k. It is also evident that the fused estimate $\hat{\delta}_F$ is always better than all the individual estimators $\hat{\delta}_k$ and for $\delta=0.1$, its performance is quite close to the CRB.

Experiment #2: The second set of results compares the proposed estimators $\hat{\delta}_1$ and $\hat{\delta}_F$ with the estimators of Candan [3] and Aboutanios and Mulgrew [9] under the condition $\delta=0.4$. As aforementioned, the suggested estimator(s) can be used iteratively (as described in Table 1) to improve the performance. Fig. 2 shows the results of the first two iterations for these estimators. From Fig. 2(a), it can be observed that the estimator $\hat{\delta}_F$ and the Aboutanios and Mulgrew estimator yield very similar performances in

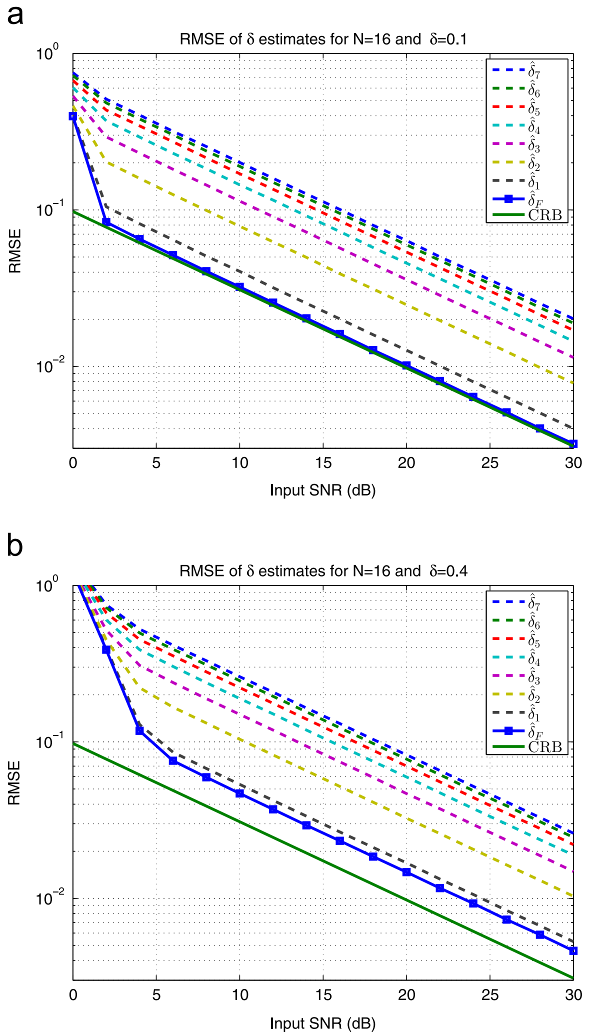

Fig. 1. RMSE performances of the estimators $\hat{\delta}_k$ and $\hat{\delta}_F$. (a) True $\delta = 0.1$ and (b) true $\delta = 0.4$.

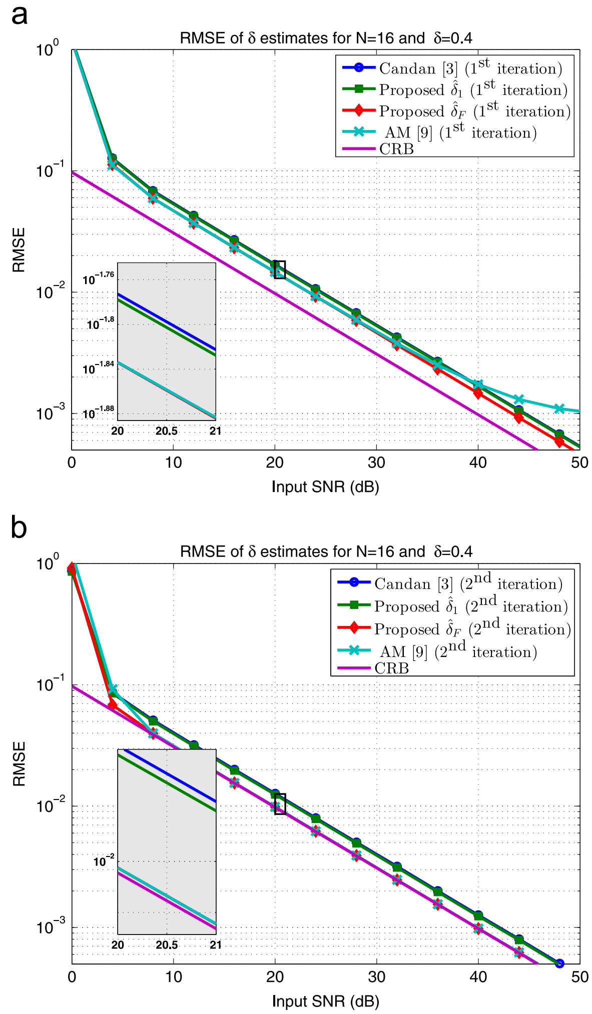

Fig. 2. RMSE performances of Candan [3], Aboutanios and Mulgrew (AM) [9] and the proposed estimators $\hat{\delta}_1$ and $\hat{\delta}_F$. (a) First iterations of the estimators and (b) second iterations of the estimators.

the first iteration; but the Aboutanios and Mulgrew estimator suffers from estimation bias at high SNR values. All four estimators operate at a non-negligible SNR gap from CRB in their first iteration. Fig. 2(b) shows the RMSE performances for the second iteration. It can be noted that the estimator $\hat{\delta}_F$ and the Aboutanios and Mulgrew estimator close the SNR gap significantly in their second iteration, while Candan's estimator and $\hat{\delta}_1$ still operate at a distance from CRB.

Experiment #3: The last set studies the SNR gaps (to the CRB) of the estimators examined in Experiment-#2. Fig. 3(a) and (b) shows the SNR gaps of the first and second iterations of the estimators, respectively, for various true δ-values under the condition SNR = 30 dB. In the first iteration, the performances of all methods are dependent on the true δ-value. It can be seen that the estimator pairs $\hat{\delta}_1$ – Candan and $\hat{\delta}_F$ – Aboutanios and Mulgrew have a similar RMSE performance. In the second iteration, RMSE has almost no dependence on the true δ-value for all methods and it is seen that the proposed estimators $\hat{\delta}_1$ and $\hat{\delta}_F$ are marginally better than their counterparts in the second iteration. It is also evident that iterating $\hat{\delta}_F$ and Aboutanios and Mulgrew estimators leads to a significant error reduction while iterating $\hat{\delta}_1$ and Candan's estimator is much less effective.

5. Conclusions

This letter presents a new estimator which uses all the bins in the DFT spectrum for the frequency estimation. The resultant estimator has an improved performance in comparison to existing estimators in the literature. The fusion technique used in this work for fusing several estimators corresponding to different DFT bins has a potential to improve other estimators if their extensions to additional DFT bins can be made.

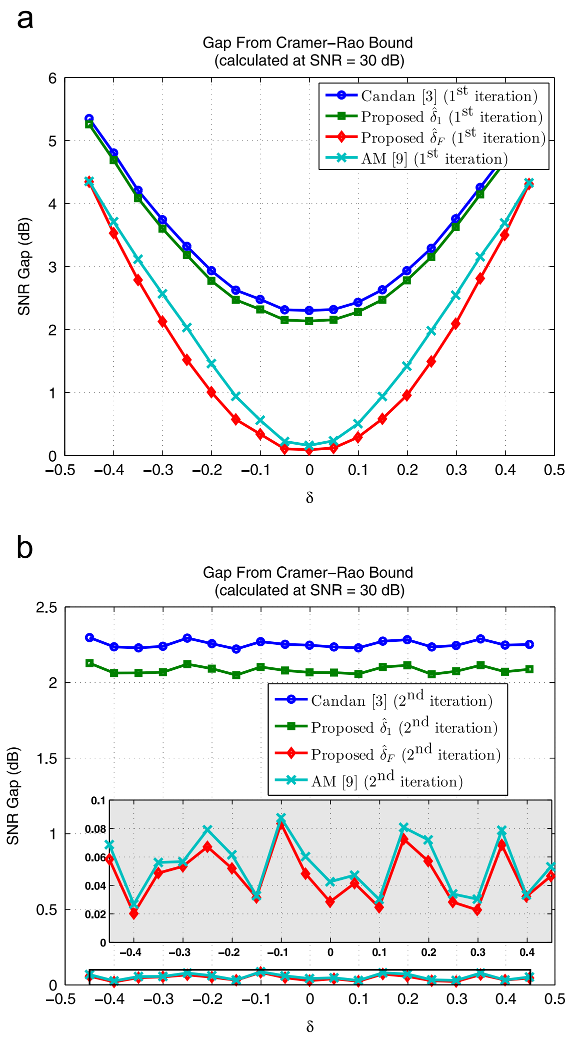

Fig. 3. SNR gaps of Candan [3], Aboutanios and Mulgrew (AM) [9] and the proposed estimators $\widehat{\delta}_1$ and $\widehat{\delta}_F$ from Cramer–Rao bound. (a) First iterations of the estimators and (b) second iterations of the estimators.

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