Digital Wideband Integrators with Matching Phase and Arbitrarily Accurate Magnitude Response (Extended Version)

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Abstract

A new class of linear phase, infinite impulse response digital wideband integrators based on the numerical integration rules is presented. Different from similar integrators in the literature, the proposed integrators exactly match the desired phase response of the continuous-time integrator (after group delay compensation) and can approximate the magnitude response as closely as desired by increasing the number of system zeros of the system, which is called the order of the system. The low order integrators (up to 4th degree) generated by this technique can be immediately utilized in many applications such as strapdown inertial navigation systems, sampled data systems and other applications, especially in control area, which require long term integration.

Index Terms

Digital Integrators, Numerical Integration, Quadrature, Newton-Cotes, Lagrange Interpolation.

I. Introduction

The digital integrators are utilized in many applications including the navigation and control systems for which a high degree of accuracy can be required, [1]. In this paper, we present a new class of discrete-time, infinite impulse response filters whose frequency response approximates the frequency response of the continuous-time integrator as accurately as it is desired, that is

$$H(e^{j\omega}) \approx \frac{1}{j\omega} = D(e^{j\omega}), \quad \text{where} -\pi < \omega < \pi.$$

In the equation shown above, $H(e^{j\omega})$ denotes the digital integrator; $D(e^{j\omega})$ denotes the desired response.

Several digital integrators designs have been proposed in the literature,[2], [3], [4], [5], [6], [7], [8], [9]. Among these the zero-order hold[1], $H(z) = \frac{1}{1-z^{-1}}$, and the trapezoidal rule, $H(z) = \frac{(1+z^{-1})}{2(1-z^{-1})}$, are the simplest and the most well known approximations to the desired response. It can be easily observed that as ω approaches zero, $H(e^{j\omega}) \rightarrow \frac{1}{j\omega}$ for both of these approximations. These approximations can be good enough for the over-sampled signals (signals sampled much above the Nyquist rate), but a closer inspection shows that there is still room for better designs especially when the signals are critically sampled at the Nyquist rate. In this paper, we present a method exactly matching the phase response of the continuous-time integrator and closely approximating the desired magnitude response. Our approach can be considered as an adaptation of the numerical integration (quadrature) rules to the integrator design problem.

Recently, Ngo and Tseng have suggested the usage of quadrature rules, such as the Newton-Cotes rule and the Gauss-Legendre rule, for the digital integrator design, [4], [5], [6]. The integrators based on the Newton-Cotes rules can be expressed in the following general form, [4]:

$$H(z) = G_N(z)\frac{1}{1-z^{-1}} \tag{1}$$

$G_N(z)$ appearing in the equation above is an Nth order causal, finite impulse response (FIR) filter. Previously discussed the zero-order hold and the trapezoidal rule filters can be put in this form by selecting $G_N(z)$ appropriately, [4].

The Newton-Cotes quadrature is based on finding the Nth order polynomial passing through $N+1$ consecutive input samples (the Nth order Lagrange interpolator) and then calculating the area under the interpolating polynomial. In Figure 1, an example with the 4th order Lagrange interpolator is given. The calculated area in the conventional system is the area between the samples $(k-1)$ and k (shown with the gray shading), [4]. The task of $G_N(z)$ filter is to implement the mentioned area calculation from $N+1$ neighboring data samples. The term following $G_N(z)$ on the right hand side of (1), which is $\frac{1}{1-z^{-1}}$, is the conventional accumulator which is used to sum the area strips calculated up to that instant, [4].

After the work of Ngo [4], Tseng has extended and presented improved techniques using similar quadrature rules in [5], [6]. Unfortunately the method of Tseng requires fractional sampling rates complicating both the design and the implementation of these integrators. Tseng suggests to use the Lagrange interpolators to elevate some of these problems. The most recent proposal by Tseng et al. suggests to implement the fractional delays in the Hartley transform domain, [8]. A proposal by Pei et al. presents discrete approximations for various continuous-time operators including differentiators

[1]Throughout the paper, we assume that the continuous-time signal is sampled at the rate of 1 samples per second. The integrators discussed should be multiplied by the sampling period T, if $T \neq 1$.

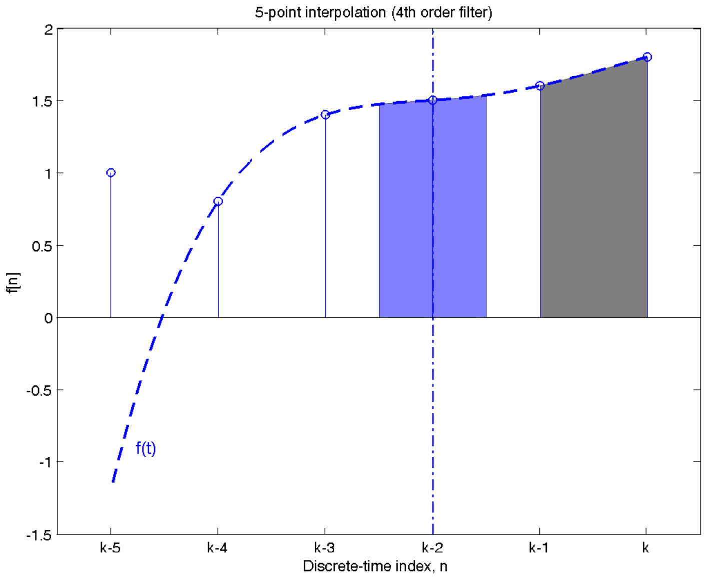


Fig. 1. 4th Order Lagrange Interpolator

and integrators in [9]. The work of Pei is based on matching the truncated Taylor series of the continuous-time operators with the discrete-time filters. In this paper, our goal is to present digital integrators using simple quadrature rules, therefore we do not focus on fractional delays or transform domain techniques.

In this paper, we present a modification on the integrators based on the Newton-Cotes rule. We suggest to calculate the area of the center strip (shown with blue shading) instead of the most recent strip as done in [4]. We show that this simple modification improves the accuracy of the magnitude response significantly and leads to a perfect match of the phase response.

In [5], Tseng has presented a general theory of integrators based on the numerical integration rules. It has been recognized during the review stage of this paper that the $G_N(z)$ polynomial of the integrators discussed here is equivalent to $U_M(z)$ polynomial given in [5], if M and L appearing in the equations (39)-(41) of [5] are taken as $M = N$ and $L = (N-1)/2$. In spite of this connection, the integrators presented here (whose performance is better than similar ones in the literature) are not explicitly known in the literature. To the best of our understanding, the parameter L appearing in [5] is considered to be an integer delay in [5] to causally implement the required fractional delays. In the present work, we present a different derivation for a sub-class of the general theory given by Tseng in [5] and show that the presented sub-class surpasses the suggested integrators in the same work and also surpasses more recent proposals, [6].

The paper is organized as follows: In the following section, we explain our design motivation and then present a derivation for area calculating $G_N(z)$ filter for even-odd values of N and present the general formulation in a compact form. Then we compare the proposed integrators with the ones in the literature and finally conclude with remarks and some application notes.

II. Proposed Integrators

We start with a simple discussion to better illustrate our motivation. An implementable digital integrator system in the form of (1) should a have right-sided $G_N(z)$. If $G_N(e^{j\omega})$ is the response of a causal and linear phase filter, then $G_N(e^{j\omega})$ can be written as $G_N^{zp}(e^{j\omega})e^{-j\frac{N}{2}\omega}$. Here N is the order of the filter and $G_N^{zp}(e^{j\omega})$ is the zero phase, or constant phase, version of $G_N(e^{j\omega})$. Under these conditions, the frequency response of the integrator can be written as follows:

$$H(e^{j\omega}) = \frac{G_N^{zp}(e^{j\omega})e^{-j\frac{N}{2}\omega}}{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})} = \frac{G_N^{zp}(e^{j\omega})}{2j\sin(\frac{\omega}{2})}e^{-j\omega(N-1)/2}. \quad (2)$$

This relation shows that a causal and linear phase $G_N(z)$ can only approximate the integrator with a delay of $(N-1)/2$ samples. It can be easily checked that among the four types of linear phase systems, only the Type-1 and Type-2 linear phase systems with symmetric impulse responses ($h[n] = h[N-n]$) are suitable for the approximation of the integrator, [10, p.257].

The main idea of the paper is to impose Type-1 or Type-2 symmetry conditions on $G_N(z)$. To do that, we propose to calculate the area under the central strip (shown with blue shading in Figure 1) instead of the strip conventionally calculated with Newton-Cotes rule (shown with gray shading) [4]. The reason of this choice can be explained as follows. It can be noted that the blue strip contains an equal number of samples on its left and right side. If the samples on the left and right side are interchanged, that is the samples on each side are flipped to the other side, the interpolation curve flips to the other side; but the area under the center strip does not change. This shows that the $G_N(z)$ filter which calculates the area of the center strip should be a symmetric polynomial. In the rest of this paper, we pursue this idea and derive Type-1 and Type-2 linear phase $G_N(z)$ functions. The derivation is given separately for Type-1 (even N) and Type-2 (odd N) for the sake of clarity.

A. *Proposed System with Even Order Interpolators*

We present a derivation for $G_N(z)$ based on the discrete-time Taylor series. To introduce the discrete-time Taylor series, we first review the difference operators and factorial polynomials[2].

The factorial polynomial $t^{[N]}$ is an Nth degree polynomial, $t^{[N]} = t(t+1)\dots(t+N-1)$. The backward difference operator Δ is defined as $\Delta f[n] = f[n] - f[n-1]$. When the backward

[2]Further discussions on the Lagrange interpolation and the discrete-time Taylor series, within the context of fractional delay systems, can be found in [11].

difference operation is executed on the factorial polynomials, we get $\Delta t^{[N]} = N t^{[N-1]}$. Hence with the application of the difference operator, we get a factorial polynomial with one less degree, in complete analogy with the continuous time derivative operator and t^N polynomial.

The discrete-time Taylor series, in analogy with its continuous version, is defined as follows:

$$f(t) \;=\; \sum_{n=0}^{\infty} \Delta^n f[k] \frac{(t-k)^{[n]}}{n!} \tag{3}$$

It can be easily noted that $f(t)$ is identical to $f[k]$ when $t = k$. If Δ is applied to both sides of (3) and then t is replaced with k; we get $\Delta f(t) \downarrow_{t=k} = \Delta f[k]$. By repeating this operation, we can show that $f(k-q) = f[k-q]$ for $q \geq 0$. This shows that $f(t)$ is an interpolating polynomial.

When the discrete-time Taylor series is truncated to a finite number of terms, the resultant relation is equivalent to fitting an Nth degree polynomial to the samples $f[k], f[k-1], \ldots, f[k-(N-1)]$, which is the Lagrange interpolation, [11].

Figure 1 illustrates the case for the 4th order interpolation. We can explicitly write the relation for this case as follows:

$$f(t) \;=\; f[k] + \frac{\Delta f[k]}{1!}(t-k) + \frac{\Delta^2 f[k]}{2!}(t-k)^{[2]} + \frac{\Delta^3 f[k]}{3!}(t-k)^{[3]} + \frac{\Delta^4 f[k]}{4!}(t-k)^{[4]} \tag{4}$$

The Newton-Cotes formula emerges immediately when $f(t)$ given in (4) is integrated between $k-1$ and k. Due to the reasons explained before, we are interested in the area under the center strip which is the integral of $f(t)$ between $k-(N+1)/2$ and $k-(N-1)/2$. The strip for $N = 4$ is shown in Figure 1.

$$\int_{k-(N+1)/2}^{k-(N-1)/2} f(t)dt = \int_{-1/2}^{1/2} f(t+k-N/2)dt = \sum_{n=0}^{N} \Delta^n f[k] \int_{-1/2}^{1/2} \frac{(t-N/2)^{[n]}}{n!} dt \tag{5}$$

Once the integrals are evaluated, the area under the center strip shown in Figure 1 can be written as follows:

$$\text{Area} = f[k] + \Delta f[k]\,(-2) + \Delta^2 f[k]\left(\frac{25}{24}\right) + \Delta^3 f[k]\left(\frac{-1}{24}\right) + \Delta^4 f[k]\left(\frac{17}{5760}\right) \tag{6}$$

When the backward difference operators are replaced with their z-domain counterparts, $\Delta^L f[k] \leftrightarrow (1-z^{-1})^L F(z)$; we get the following system function for the area calculator as follows:

$$G_4(z) = 1 + (1-z^{-1})\,(-2) + (1-z^{-1})^2\left(\frac{25}{24}\right) + (1-z^{-1})^3\left(\frac{-1}{24}\right) + (1-z^{-1})^4\left(\frac{17}{5760}\right) \tag{7}$$

$$= z^{-2} \underbrace{\left(-\frac{17}{5760}(z^2+z^{-2}) + \frac{77}{1440}(z^1+z^{-1}) + \frac{863}{960}\right)}_{G_4^{zp}(z)} \tag{8}$$

As desired, $G_4(z)$ is a Type-1 linear phase sequence. A zero-phase sequence, shown as $G_4^{zp}(z)$, can be constructed from $G_4(z)$ by shifting it 2 units to the left (advancing). In the general case, the $G_N^{zp}(z)$ can be constructed by advancing $G_N(z)$ with $N/2$ samples.

The frequency response of the Nth order integrator, with the form $H(z) = \frac{G_N(z)}{1-z^{-1}}$, can be expressed as follows:

$$H(e^{j\omega}) = \frac{G_N(e^{j\omega})}{2j\sin(\frac{\omega}{2})}e^{j\frac{1}{2}\omega} = \frac{G_N^{zp}(e^{j\omega})}{2j\sin(\frac{\omega}{2})}e^{-j\frac{N-1}{2}\omega} \tag{9}$$

When N is taken as 4 in the last relation, it is evident that the system is a linear phase system with a constant group delay of $(4-1)/2 = 3/2$ samples. In other words, the kth *output sample* of the 4th order integrator sums the area up to $k - 3/2$ samples, which is exactly the scenario shown in Figure 1.

The part of frequency response given in (9) without the linear phase term is an approximation to the ideal integrator, $\int_{-\infty}^{t} f(\tau)d\tau$. The nature of the approximation for the 4th order case can be shown as follows:

$$\begin{aligned}\frac{G_4^{zp}(e^{j\omega})}{2j\sin(\frac{\omega}{2})} &= \frac{\frac{863}{1920} + \frac{77}{1440}\cos(\omega) - \frac{17}{5760}\cos(2\omega)}{j\sin(\frac{\omega}{2})} \\ &= \frac{\frac{1}{2} - \omega^2\frac{1}{48} + \omega^4\frac{1}{3840} + \omega^6\frac{13}{69120} + O(\omega^8)}{j(\omega\frac{1}{2} - \omega^3\frac{1}{48} + \omega^5\frac{1}{3840} - \omega^7\frac{1}{645120} + O(\omega^9))} \\ &\approx \frac{1}{j\omega}\end{aligned} \tag{10}$$

In the second line of (10), the Taylor series expansions of cosine and sine functions are written for the numerator and denominator terms. This result shows that the denominator polynomial in ω is equal to the numerator polynomial times $j\omega$ up to the 8th degrees. It is evident that a higher order approximation would improve the approximation accuracy to the continuous-time integrator. In the numerical results section, we examine the accuracy of this approximation for various values of N.

B. Proposed System with Odd Order Interpolators

Figure 2 shows the Lagrange interpolator and the center-strip for the 5th order interpolation. The Lagrange interpolator for the 5th order case contains all the terms on the right hand side of equation (4) and the additional term of $\frac{\Delta^5 f[k]}{5!}(t-k)^{[5]}$. The area under the center-strip can be calculated by integrating $f(t)$ between $k-(N+1)/2$ and $k-(N-1)/2$, as in the even order case. Repeating the steps described, we can get the Type-2 symmetric $G_5(z)$ polynomial as follows:

$$G_5(z) = z^{-2.5}\left(\frac{11}{1440}(z^{2.5}+z^{-2.5}) - \frac{31}{480}(z^{1.5}+z^{-1.5}) + \frac{401}{720}(z^{0.5}+z^{-0.5})\right) \tag{11}$$

The 5th order approximation for $\frac{1}{j\omega}$ is then as follows:

$$\begin{aligned}\frac{G_4^{zp}(e^{j\omega})}{2j\sin(\frac{\omega}{2})} &= \frac{\frac{11}{1440}\cos(2.5\omega) - \frac{31}{480}\cos(1.5\omega) + \frac{401}{720}\cos(0.5\omega)}{j\sin(\frac{\omega}{2})} \\ &= \frac{\frac{1}{2} - \omega^2\frac{1}{48} + \omega^4\frac{1}{3840} - \omega^6\frac{437}{276480} + O(\omega^8)}{j(\omega\frac{1}{2} - \omega^3\frac{1}{48} + \omega^5\frac{1}{3840} - \omega^7\frac{1}{645120} + O(\omega^9))} \\ &\approx \frac{1}{j\omega}\end{aligned} \tag{12}$$

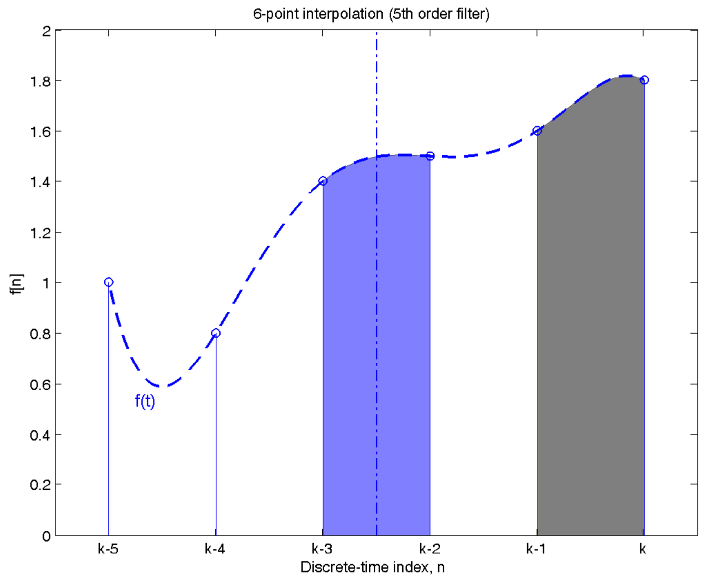


Fig. 2. 5th Order Lagrange Interpolator

It can be seen that the frequency response of the odd order integrators is identical to the response of even order integrators given in (9). The only difference is that even degree integrators have Type-1 symmetry for $G_N(z)$, while odd degree integrators have Type-2 symmetry. The group delay of both cases is $(N-1)/2$, exactly matching the illustrations given in Figures 1 and 2.

The presented examples can be generalized to all orders without any difficulty. For an arbitrary value of N, the proposed digital integrators can be written as $H(z) = G_N(z)/(1-z^{-1})$ where

$$G_N(z) = \sum_{n=0}^{N} \frac{(1-z^{-1})^n}{n!} \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{\prod_{m=0}^{n-1} (t - N/2 + m)}_{(t-\frac{N}{2})^{[n]}} dt \tag{13}$$

The expressions for $G_N(z)$ up to the 7th order and the Matlab code generating an arbitrary order $G_N(z)$ are presented in Table I.

III. Numerical Comparisons and Implementation Issues

In this section, we present a numerical comparison of the proposed integrators with the ones in the literature. The first proposal on digital integrators utilizing Newton-Cotes rules has been given by Ngo in [4]. Recently similar integrators have been proposed by Tseng, [6]. The integrators proposed by Tseng have better performance when compared to other integrators requiring similar amount of computation. We choose to compare the proposed integrators with the classical Ngo integrator and recently proposed Tseng integrators [6] and the Simpson integrator [7]. (Interested readers can

TABLE I

PROPOSED DIGITAL INTEGRATORS, $H(z) = G_N(z)/(1-z^{-1})$.

N	$G_N(z)$
1	$\frac{1}{2}+\frac{1}{2}z^{-1}$
2	$\frac{1}{24}(1+z^{-2})+\frac{11}{12}z^{-1}$
3	$-\frac{1}{24}(1+z^{-3})+\frac{13}{24}(z^{-1}+z^{-2})$
4	$-\frac{17}{5760}(1+z^{-4})+\frac{77}{1440}(z^{-1}+z^{-3})+\frac{863}{960}z^{-2}$
5	$\frac{11}{1440}(1+z^{-5})-\frac{31}{480}(z^{-1}+z^{-4})+\frac{401}{720}(z^{-2}+z^{-3})$
6	$\frac{367}{967680}(1+z^{-6})-\frac{281}{53760}(z^{-1}+z^{-5})+\frac{6361}{107520}(z^{-2}+z^{-4})+\frac{215641}{241920}z^{-3}$
7	$-\frac{191}{120960}(1+z^{-7})+\frac{1879}{120960}(z^{-1}+z^{-6})-\frac{353}{4480}(z^{-2}+z^{-5})+\frac{68323}{120960}(z^{-3}+z^{-4})$

Matlab Script Generating $G_N(z)$ Polynomials

```
order=5; %set the order
syms t zi; G=1; %init
for k=1:order,
   dum=prod(t-order/2:(t-order/2+k-1))/prod(1:k);
   G=G+int(dum,-1/2,1/2)*(1-zi)^k;
end;
simplify(G),
```

examine Figure 6 of [6] for a comparison of Tseng's integrators with the earlier integrators in the literature.)

The definitions for Ngo, Tseng and Simpson integrators are given as follows:

$$H_N(z) = \frac{9+19z^{-1}-5z^{-2}+z^{-3}}{24(1-z^{-1})} \tag{14}$$

$$H_T(z) = \frac{-3693+67260z^{-1}+88650z^{-2}-14388z^{-3}+2139z^{-4}}{139968(1-z^{-1})}z \tag{15}$$

$$H_S(z) = \frac{1+4z^{-1}+z^{-2}}{3(1-z^{-2})} \tag{16}$$

Different from Ngo and Tseng integrators, the integrator based on the Simpson's rule has two poles at $z=\{1,-1\}$ and has a symmetric numerator polynomial leading to a perfect phase match with the ideal integrator. The Simpson integrator and its improved versions suggested by Tseng can be found in [7]. We note that the integrators whose poles are uniformly distributed around the unit circle, such as the Simpson integrator, can also be expressed in the framework set in [4].

Frequency Response Comparison: Figure 3 shows the magnitude response of the proposed integrators and the ideal integrator. The next figure shows the approximation error magnitude, $D(e^{j\omega}) - H(e^{j\omega})$ which is the deviation from the desired response. Here $D(e^{j\omega}) = 1/(j\omega)$ is the response of the ideal integrator.

For the proposed integrators presented in Figures 3 and 4, the group delay values are different from each other. For comparison purposes, the group delay of each integrator is compensated making each system constant phase. More specifically, the linear phase term appearing on the right-most side of (9) is removed so that the response approximates $\frac{1}{j\omega}$ as shown in (10).

It can be seen from Figures 3 and 4 that the proposed integrators, especially the ones with even orders, perform better than earlier proposals. The proposed 2nd order integrator is better than Ngo

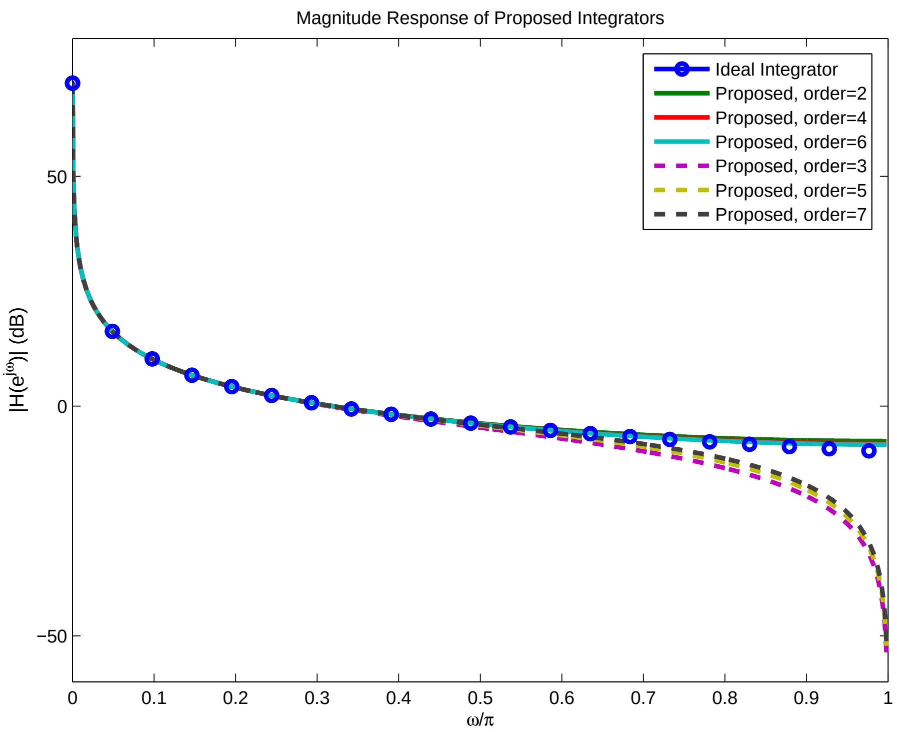


Fig. 3. The magnitude response of proposed integrators and the ideal integrator.

integrator (which is 3rd order) and marginally better than Tseng integrator (which is 4th order). The performance gap widens for the proposed 4th order integrator.

Figures 3 and 4 show that the integrators with even orders significantly outperform the odd ordered ones at high frequencies. This is due to the structure of Type-1 and Type-2 filters. It should be remembered that a Type-2 sequence has a zero at $z = -1$, leading to $G_{2L+1}(e^{j\omega}) \to 0$ as $\omega \to \pi$. On the other hand, the ideal integrator $(\frac{1}{j\omega})$ approaches $1/(j\pi)$ as $\omega \to \pi$. The presence of a zero at $z = -1$ results in an undesired attenuation of the response at high frequencies leading to observed performance gap between even and odd orders.

Figure 5 compares the phase response of digital integrators. The proposed integrators, by design, attain the desired phase of -90 degrees at all frequencies. The phase of -90 degrees is attained after group delay compensation, which is $\frac{N-1}{2}$ samples. For even ordered integrators, this leads to a half sample delay as can be seen from Figure 1. It should be noted that Ngo and Tseng integrators significantly deviate from the desired response in mid frequency bands. This deviation is expected since the numerator polynomials given in the corresponding equations of (14) are not symmetric. On the other hand, the Simpson integrator has a perfect phase match due to its symmetric numerator polynomial.

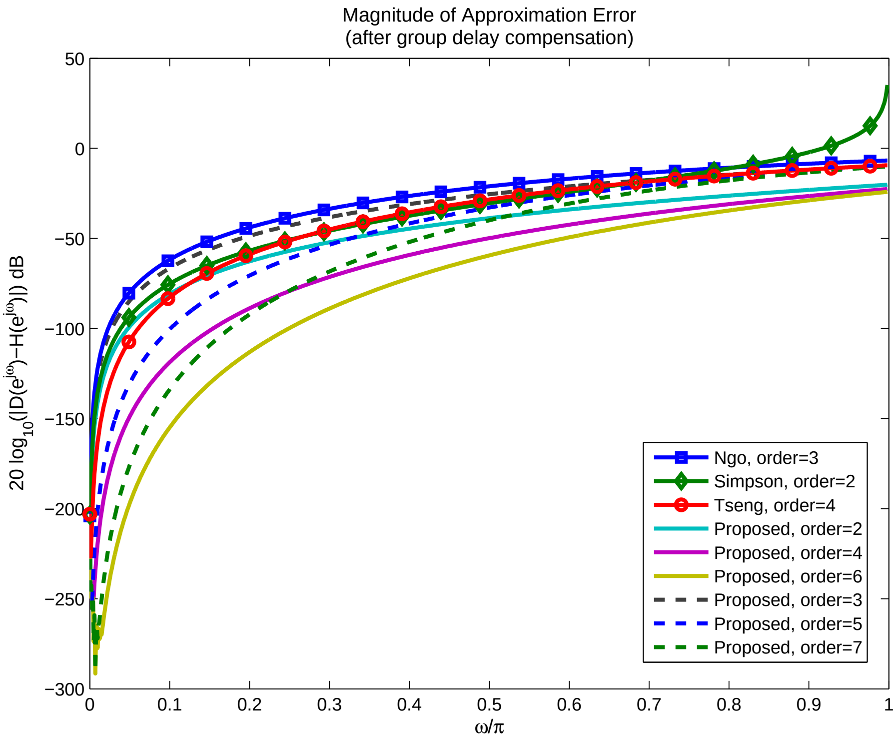


Fig. 4. Magnitude Spectrum of Approximation Error

An Application Example: A typical navigation system contains a set of accelerometers and gyroscopes on the moving body to sense the changes in the position and the orientation of the moving object, [1]. It is well known that the second integral of the accelerometer readings is required to calculate the instantaneous position of the object. Similarly, the first integral of the rate gyroscopes is required to calculate the instantaneous orientation of the object. In this application example, we place a rate gyroscope in the test unit shown in Figure 6(a) and program the unit to make rotations about a single axis. The instructed motion is not the conventional uniform circular motion, but has intervals of acceleration and de-acceleration as can be seen from the gyroscope readings given in Figure 6(b). (The data shown in Figure 6(b) is collected with Sensonor STIM 202 gyro at the sampling rate of 1000 samples per second.)

As seen from Figure 6(b), the sequence of time intervals having acceleration and de-acceleration motion results in an almost periodic gyro readings whose period is measured as 950 samples. The change in the heading angle of the object can be found by integrating the data shown in this figure. For signals of such low frequencies, i.e. $\omega = \frac{2\pi}{950}$, the digital integrators that are compared in this paper produce almost identical results. This result is not surprising in the light of previous findings given in Figure 4.

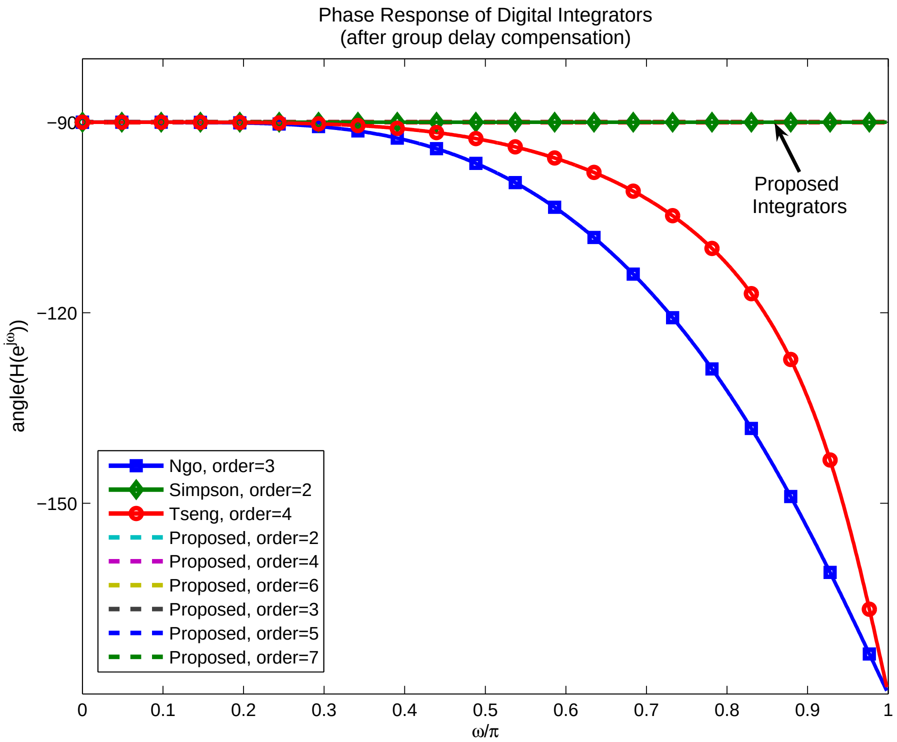


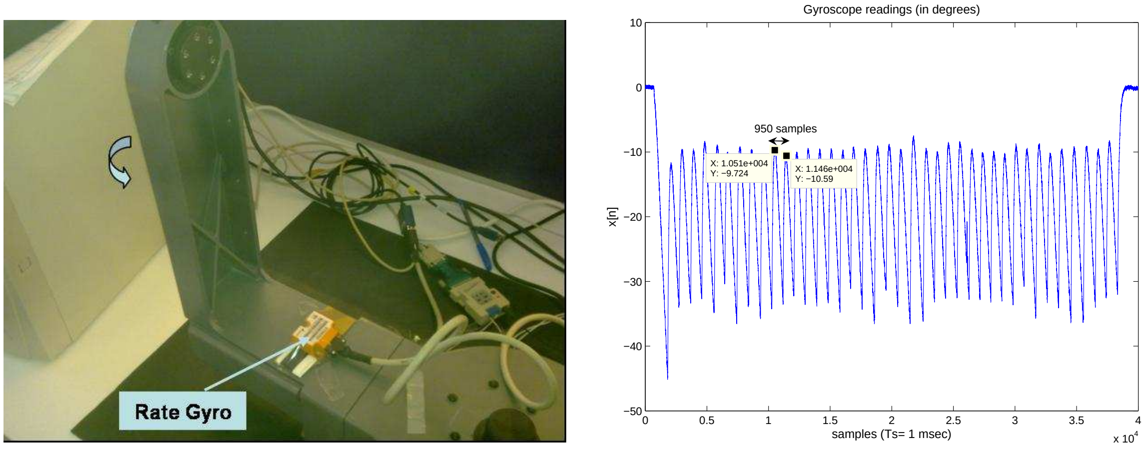
Fig. 5. Phase Response of Digital Integrators

To compare the performance of digital integrators, we decimate the collected the data through the system shown in Figure 7. The downsampled data has a period of $\frac{950}{L}$ where L is the downsampling ratio, which, in effect, leads to a increase in the frequency of periodic discrete-time signal.

In the conducted experiment, the outputs of the digital integrators are compared at different downsampling rates. The ground-truth value is taken as the final value of the integration for the undecimated ($L = 1$) system. It should be noted for the undecimated system, all integrators produce the same output up to 3 decimal digits. (The final value is 738.417 degrees.) The deviation from the final value is considered as the integration error.

Figure 8 shows the variation of the error expressed in decibels at different downsampling rates. It can be observed that the proposed fourth order integrator produces the best result up to the downsampling ratio of 5. For higher downsampling ratios, the performance of fourth order system is almost identical to the third order system and to the system proposed by Tseng. It can be noted that the results of the experiment are in agreement with the earlier findings presented in Figure 4.

It is important to note that for the navigation application, the integration error is one of the many contributors to the final error. The other factors such as gyroscope bias, scaling factor, stability and the errors in its alignment can have significantly more impact on the overall performance, [1]. With



(a) Experimental Set-up

(b) Gyroscope measurements

Fig. 6. Set-up for the gyroscope testing and collected measurements

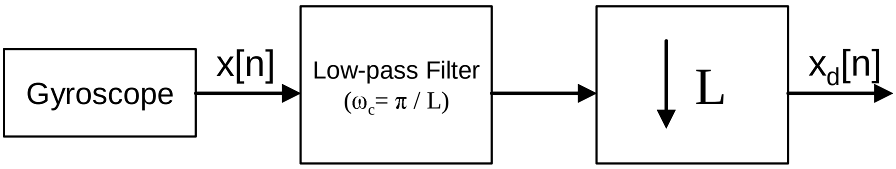


Fig. 7. The down-sampling system utilized in the application demonstration

this experiment, our goal is to isolate the integration error by generating the experimental data from a single run so that the run-time errors and time-varying bias is kept constant in all comparisons. In practice, the error due to the integrator is a minor component in comparison to other factors for the non-tactical grade gyros. In spite of this fact, the proposed integrators does not require any more computational resources than its alternatives and therefore can only bring gain which can be especially important in high accuracy systems.

Implementation Issues: The simplest digital integrator is the cumulative summer whose system function is $1/(1-z^{-1})$. The cumulative summer does not require any multipliers for its implementation. The proposed integrators and the other integrators residing in the same class have the form of $G_N(z)/(1-z^{-1})$. It can be easily noted from (14) that Tseng and Ngo integrators require N multipliers per output sample where N is the order of filter. As can be seen from Table I, the proposed integrators require $\lceil \frac{N+1}{2} \rceil$ multipliers per output sample due to their even symmetry. (Here $\lceil \cdot \rceil$ is the function mapping the argument to the smallest integer greater than or equal to the argument.)

It can be noted that the fourth order integrator proposed in this paper has computational requirements equivalent to the Ngo integrator and has one less multiplier requirement than the Tseng integrator; but it produces a better performance than both alternatives. We believe that for many applications, there would be little return for using fifth or higher order integrators; hence the second order or the fourth order integrator can suffice in many applications.

It should be noted that the integrator, by definition, is an unstable system, therefore it is only applicable for inputs having zero mean. Even for a zero mean input, the word width of the integrator should be sufficiently large to accurately hold the accumulation result. For the application example discussed, the angles produced by the integrators can be reduced to their modulo 2π equivalents whenever it is appropriate. As a final note, two's complement (non saturating) arithmetic can be adopted to aid the recovery from a potential overflow.

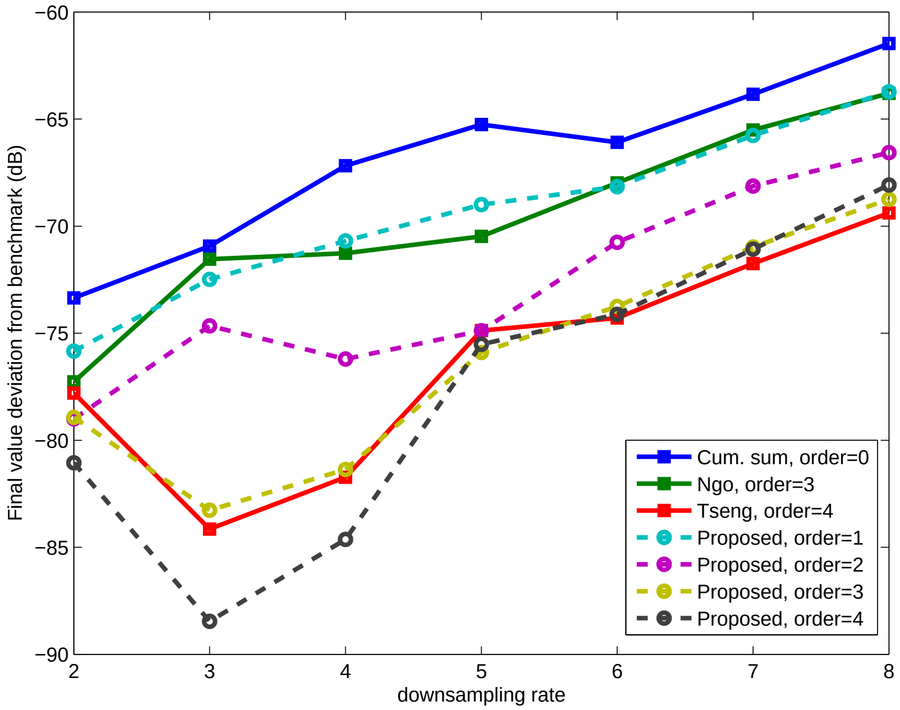


Fig. 8. Comparison of digital integrators

IV. Conclusions

In this paper a class of digital integrators has been described. By design, the digital integrators match the phase response of the continuous-time integrator and can approximate the magnitude response with an arbitrary degree of accuracy across a wide-band of frequencies. The second and fourth order integrators, which require little computation per output sample, can immediately replace earlier proposals in many applications requiring long term integration such as navigation applications, [1].

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References

[1] E. Bekir, *Introduction to Modern Navigation Systems*. World Scientific Publishing, 2007.

[2] B. Kumar, D. Choudhury, and A. Kumar, "On the design of linear phase, FIR integrators for midband frequencies," *IEEE Trans. Signal Process.*, vol. 44, no. 10, pp. 2391–2395, 1996.

[3] M. Al-Alaoui, "Novel digital integrator and differentiator," *Electronics Letters*, vol. 29, no. 4, pp. 376–378, 1993.

[4] N. Q. Ngo, "A New Approach for the Design of Wideband Digital Integrator and Differentiator," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 53, no. 9, pp. 936–940, 2006.

[5] C.-C. Tseng, "Closed-Form Design of Digital IIR Integrators Using Numerical Integration Rules and Fractional Sample Delays," *IEEE Transactions on Circuits and Systems I*, vol. 54, no. 3, pp. 643–655, 2007.

[6] C.-C. Tseng and S.-L. Lee, "Digital IIR Integrator Design Using Richardson Extrapolation and Fractional Delay," *IEEE Transactions on Circuits and Systems I*, vol. 55, no. 8, pp. 2300–2309, 2008.

[7] C.-C. Tseng, "Digital integrator design using Simpson rule and fractional delay filter," *IEE Proceedings - Vision Image and Signal Processing,*, vol. 153, pp. 79–86, Feb. 2006.

[8] C.-C. Tseng and S.-L. Lee, "Design of digital IIR integrator using discrete Hartley transform interpolation method," *IEEE International Symposium on Circuits and Systems*, pp. 2181–2184, May 2009.

[9] S.-C. Pei, P.-H. Wang, and C.-H. Lin, "Design of Fractional Delay Filter, Differintegrator, Fractional Hilbert Transformer, and Differentiator in Time Domain With Peano Kernel," *IEEE Transactions on Circuits and Systems I*, vol. 57, no. 2, pp. 391–404, 2010.

[10] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*. Prentice-Hall, 1989.

[11] C. Candan, "An Efficient Filtering Structure for Lagrange Interpolation," *IEEE Signal Process. Letters*, vol. 14, no. 1, pp. 17–19, 2007.