THE DISCRETE FRACTIONAL FOURIER TRANSFORM

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this work, the discrete counterpart of the continuous Fractional Fourier Transform (FrFT) is proposed, discussed and consolidated. The discrete transform generalizes the Discrete Fourier Transform (DFT) to arbitrary orders, in the same sense that the continuous FrFT generalizes the continuous time Fourier Transform. The definition proposed satisfies the requirements of unitarity, additivity of the orders and reduction to DFT. The definition proposed tends to the continuous transform as the dimension of the discrete transform matrix increases and provides a good approximation to the continuous FrFT for the finite dimensional matrices. Simulation results and some properties of the discrete FrFT are also discussed.

Keywords: Fractional Fourier Transform, Discrete Fourier Transform, Discrete Fractional Fourier Transform

ÖZET

AYRIK KESİRLİ FOURIER DÖNÜŞÜMÜ

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Bu çalışmada ayrık kesirli Fourier dönüşümü önerilmiş ve incelenmiştir. Önerilen tanım, sürekli kesirli Fourier dönüşümünün sürekli Fourier dönüşümünü genellediği şekilde, ayrık Fourier dönüşümünü istenilen herhangi bir dereceye geneller. Önerilen tanım, birimcil olma, derece eklenebilirlik ve ayrık Fourier dönüşümününe sadeleşme özelliklerine sahiptir. Ayrıca bu tanımın sürekli kesirli Fourier dönüşümüne yakınsadığı da gösterilmiştir. Bu tanımın bazı özelliklerinin yanısıra benzeşim sonuçları da sunulmuştur.

Anahtar Kelimeler: Kesirli Fourier Dönüşümü, Ayrık Fourier Dönüşümü, Ayrık Kesirli Fourier Dönüşümü

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Contents

List of Figures

List of Tables

To My Parents

Chapter 1

Introduction

With the development of high power and cheap computers, digital signal processing has replaced analog or continuous time signal processing because of its exactness, versatility and feasibility. Fourier transform, one of the most important transforms in the continuous time signal processing, has found its counterpart in discrete time as the discrete Fourier transform (DFT) and DFT, along with its fast implementation, became one of the intensively used tools utilized especially in applications such as filtering, coding, modulation etc. Recently a new transform, the fractional Fourier transform (FrFT[1]) was rediscovered [1, 2, 3] independently of the previous works [4, 5]. The fractional transform generalizes the continuous Fourier Transform to arbitrary orders and reduces to the continuous Fourier Transform at the special cases. With the utilization of the additional degree of freedom, the order of the transform, FrFT has found many interesting applications, such as improved Wiener filtering [6, 7], cost efficient approximation of linear systems [8, 9], time-frequency domain analysis [10, 11, 12, 13], analysis and design of optical systems [14, 15, 16] and signal representation [17, 18, 19, 20]. The discrete counterpart of the continuous FrFT is yet an open problem, in this work we discuss and consolidate the discrete definition of FrFT.

[1] Usage of the acronym FrFT is not accepted by everyone, including the supervisor of this thesis.

In engineering applications the integral of the continuous Fourier Transform is rarely evaluated, because of its high cost of computation, but in general one approximates the samples of the Fourier Transform by taking DFT of the function to be transformed. As size of the DFT matrix increases, the DFT tends to the continuous Fourier Transform, which is a fact evident from the comparison of the kernels of the discrete and continuous transforms, leading to DFT's approximation property of the continuous Fourier Transform.[2] Apart from the approximation of the continuous Fourier Transform, the other properties of the DFT such as the ones related with cyclic convolutions, difference equations, etc. make this transform an invaluable tool for processing discrete signals.

As mentioned before, FrFT generalizes ordinary Fourier Transform to a class of transforms, which includes ordinary Fourier Transform as a special case. It is natural to expect the discrete equivalent of the FrFT to generalize the DFT to arbitrary orders and to approximate the continuous fractional Fourier transform. Unfortunately a satisfying definition for the discrete transform has not emerged until now due to the multiplicity of the possible definitions for the fractional Fourier transforms. What we mean by the multiplicity of the definitions is the possibility of the *distinct* definitions for fractional transforms generalizing the ordinary Fourier Transform. Our choice, out of infinitely many different transform possibilities, constitutes a legitimate one which has found many applications in different fields, such as quantum physics, optics and signal processing. Similarly if one only aims to generalize the DFT, there exists again infinitely many distinct definitions, but in this work we aim to find the discrete FrFT corresponding to our definition of the continuous fractional Fourier Transform.

We will now briefly review the definition and the properties of the Fractional Fourier Transform. The definition of the Fractional Fourier Transform can be given as

$$\{\mathbf{F}^a f\}(t_a) \quad = \quad A_\phi \int_{-\infty}^{\infty} e^{j\pi(t_a^2 \cot\phi - 2t_a t' \csc\phi + t'^2 \cot\phi)} f(t')dt' \qquad (1.1)$$

where $\phi = a\pi/2$ and $A_\phi = \frac{\exp(-j(\pi sgn(\sin(\phi))/4-\phi/2))}{|\sin(\phi)|^{0.5}}$. The transform reduces to the ordinary Fourier transform, $\{\mathbf{F}f\}(t_1) = \int e^{-j2\pi t_1 t'} f(t')dt'$, when $a = 1$ and

[2]The exact relation between DFT and Fourier Transform is given by Poisson's theorem. One can find a brief presentation of this theorem in the appendix.

it can be shown that the kernel approaches $\delta(t_a - t)$ and $\delta(t_a + t)$ when a approaches 0 and 2 respectively [21]. Some properties of the FrFT are

1. Unitarity.

2. Additivity of the orders ($\mathbf{F}^{a_1}\mathbf{F}^{a_2} = \mathbf{F}^{a_1+a_2}$).

3. Reduction to ordinary Fourier Transform at $a = 1$.

4. Having Hermite-Gaussian functions as eigenfunctions.

5. Rotation of Wigner distribution by $a \times 90$ degrees.

One can find more properties of the FrFT in [22, 23, 24, 11].

It is legitimate to expect from the discrete equivalent of FrFT to have some properties similar to the ones of the continuous time transform. We propose that the following requirements should be the properties of a discrete definition which is said to be analog of the continuous FrFT.

1. Unitarity.

2. Additivity of the orders.

3. Reduction to DFT.

4. Approximation of the FrFT.

The first two requirements should obviously be satisfied, if one claims to find the discrete analog of the FrFT. The third condition is required if one argues to generalize the DFT. The last requirement seeks for a relation, like Poisson's theorem, between the continuous FrFT and discrete FrFT.

We will see in the next chapters that the first three requirements are relatively easy to satisfy, but finding a correspondence between the continuous and discrete transform is the real difficulty of the problem. Before explaining our method of discretization, we will examine previous work on the problem.

1.1 Previous Work

We divide the papers related with the discretization of FrFT into 2 categories upon their nature of definition. First category includes the papers whose aim is to calculate the integral of the FrFT from samples of the input to the approximate samples of its FrFT. The other category includes the papers in which the mapping introduced not only approximates the continuous FrFT, but also satisfies some requirements of the discrete definition. We will examine both categories together, but one should remember this discrimination of the papers while reading this section.

FrFT calculation via Hermite-Gaussian series [2, 3]: In this paper, the computation of the FrFT is established by expressing the input as superposition of Hermite-Gaussians (Hermite-Gaussian series) and taking FrFT of the series. Properties of the Hermite-Gaussians, (see Chapter 2), lead to easy determination of weight and fractional Fourier transform of each Hermite Gaussian term. This method is effective for a certain class of signals, where the input can be approximated up to certain degree with the first few Hermite Gaussians. It can be seen that this class consists of signals with sufficient amount of energy around origin, therefore signals with "mean" energy far from origin, such as a shifted rectangle, can not be easily expressed with a few terms. Another drawback of this method is that the "average" of the signal (DC term of the Fourier series) is represented by all even ordered Hermite-Gaussians, therefore whenever you approximate a non-Hermite-Gaussian function with Hermite-Gaussians, there will be an error in the mean value. As a result, one can think Hermite-Gaussian series as the expansion of the signal in terms of the local values of the signal, showing the difficulty of expressing the mean. One can easily see that the sole aim of this method is calculating the FrFT without chirp integrals given in (1.1).

FrFT calculation of time and frequency limited functions [25]: This method starts with the definition of the FrFT as given in (1.1) and establishes a mapping from the samples of the input to the (approximate) samples of the output for a certain class of signals. The class discussed above is the span of signals that are both (approximately) time and frequency limited. We know that there does not exist a function that is both time and frequency limited, but

most "physical" signals can be thought as approximately time and frequency limited. The method involves two steps, first frequency band-limitedness is used to express the function using Shannon's sampling theorem, then time-limitedness is used to limit the infinite summation of sincs to a finite one (since time samples of the function can be ignored when sampling location exceeds time limit of the signal). As a result, the function is approximated by a finite sinc series and then the FrFT of each term of the series is taken. Finally when output is sampled, we reach a relation from the finite samples of the input to the samples of its FrFT. If we observe the matrix of this discrete mapping, we see that the matrix is neither unitary nor satisfies the additivity of the orders. But for the class of signals discussed, the mapping introduced is almost unitary, that is denoting the matrix of the mapping by $\mathbf{M}$, $(\mathbf{M}f, \mathbf{M}f) - (f, f) \approx 0$, for $f(t)$ in the class discussed. This method is a very powerful method of computing continuous FrFT, with its $\mathrm{O}(\mathrm{N}\log\mathrm{N})$ implementation, but it satisfies none of the requirements for the discrete definition.

FrFT calculation through sinc interpolation [26]: In this paper; authors, apparently unaware of [25], proposed a similar scheme for calculation of FrFT with finite sinc summation. Authors reach equivalent results of the previously discussed paper, but they mistakenly compare their definition with a distinct definition of the the FrFT. Some distinct definitions of FrFT and their effect on the discretization problem will be discussed in the core of the text.

FrFT calculation through FFT algorithm [27]: Authors of this paper propose a method of computation for FrFT based on chirp multiplication, chirp convolution and chirp multiplication realization of the FrFT. They replace the intermediate continuous convolution step with FFT, multiplication in FFT domain and inverse FFT. This method can be utilized successfully, only if the aliasing effects in time and frequency domains are treated carefully.

Fractional powers of DFT matrix [28]: In this paper, the author aims to find $1/p^{th}$ power of the DFT matrix saying that "Our primary goal is to demonstrate that $(DFT)^{1/p}$ can be found." Although the existence of $(DFT)^{1/p}$ can be easily seen, since DFT matrix is diagonalable (see Chapter 2), author finds examples of the $1/p^{th}$ power DFT matrix for dimensions of 2 and 4. $(DFT)^{1/p}$ matrices satisfies first three requirements, but nothing can be said for the correspondence with FrFT.

Discrete FrFT via Taylor Series [29, 30]: Both papers find the fractional powers of the DFT matrix, by expressing fractional power operation in terms of Taylor series and inserting DFT matrix into the expansion. Using the identity $\mathbf{F}^4 = \mathbf{I}$, where $\mathbf{F}$ and $\mathbf{I}$ are the DFT and identity matrix respectively, the infinite summation of the Taylor series is condensed to the summation of the four terms $(\mathbf{F}^0, \mathbf{F}^1, \mathbf{F}^2, \mathbf{F}^3)$. By further analysis authors find the fractional powers of the DFT matrix. We note that the fractional DFT matrix found by two authors are exactly the same (eqn.(26) in [29] is equivalent to eqn.(43) in [30]). The discrete fractional Fourier matrix proposed satisfies first three requirements, but we will see in the progress of this study that the discrete transform defined does not correspond FrFT defined by (1.1), but corresponds to a distinct definition of the fractional Fourier transform (see Chapter 2). Therefore discrete definition has a correspondence with continuous signals, but this correspondence is with some other distinct definition, not with the FrFT we have defined.

Discrete FrFT via Poisson's theorem [31]: As said before, Poisson's theorem enables one to determine DFT as the mapping between time and frequency domain representation of the continuous signals which are appropriately aliased and sampled. The author of this paper investigates FrFT in order to establish a similar mapping between time and ath domain representation of the signal. The author shows the existence of periodic signals both in time and ath domain when transform order a is a rational number and finds the mapping between the periodic sequences in both domains and denotes the mapping as Discrete Fractional Fourier Transform. This mapping reduces to DFT at special cases, but for the fractional cases, the parameter T_0 of the Poisson's theorem does not cancel out as in DFT case leading to the ambiguity of determination of this parameter when discrete signal has no obvious counterpart in continuous time (see [31]). This mapping is neither unitary nor satisfies the additivity of the orders.

Fractional Fourier-Kravchuk Transform [32]: In this paper, authors define a finite discrete transform using Kravchuk polynomials. Kravchuk polynomials have two interesting properties. The first one is, N samples of the first

N Kravchuk polynomials (from 0th to $(N-1)$th degree) form an orthogonal set of $\mathcal{R}^N$ with an appropriate weight function. The other one is, Kravchuk polynomials tend to Hermite-Gaussians as N increases. Since Hermite-Gaussians are eigenfunctions of the FrFT (see Chapter 3), we directly found a correspondence between FrFT and Kravchuk polynomials. By using the orthogonality of the samples of the Kravchuk polynomials, one can define a finite transform satisfying unitarity and additivity of the orders. Unfortunately this matrix does not reduce to DFT matrix at the special cases, which is the major drawback of this method. Another difficulty arises while calculating the Kravchuk polynomials of high degrees. In order to preserve the orthogonality of the sample vectors the coefficients of the high powers must be evaluated with high accuracy, which poses a computation problem.[3]

Discrete FrFT through discrete Wigner distribution [34]: In this interesting paper, authors use their definition of discrete Wigner distribution to define the discrete fractional Fourier Transform. In [35] authors define the discrete Wigner distribution using group theoretical concepts. The discrete distribution defined satisfies many properties analogous to the continuous one such as marginals, symplectic transformations etc. Authors show that the rotation of the discrete distribution is possible for a finite number of degrees and give the definition of the discrete FrFT by expressing rotation in terms of "chirp" convolution and "chirp" multiplications. We note that with this definition, it is not possible to get fractional Fourier Transform of a sequence at an arbitrary order and furthermore fractional Fourier Transform of even length sequences can not be defined. Additionally computer simulations has not revealed a correspondence with the continuous FrFT.

Discrete FrFT through "Discrete Hermite-Gaussians" [36, 37, 38]: In his first work, [36], Pei trys to construct a set of eigenvectors of DFT resembling the functional behavior of Hermite-Gaussians, which are eigenfunctions

[3] A recent work [33], unaware of [32], attempted to utilize the Kravcuk polynomials for the discrete definition. According to the author of this thesis, the paper submitted has conflicts with [32] and furthermore some claims of the paper can not be justified by computer simulations. Unfortunately, the multiplicity of definitions of FrFT is not taken into account by the author of [33].

of fractional Fourier transform (see Chapter 3). Since samples of the Hermite-Gaussians are not eigenvectors of DFT, Pei proposes a novel LMS algorithm that produces Hermite-Gaussian like eigenvectors of DFT in [36].

In his more recent works [37, 38], Pei notices that the eigenvectors of a matrix, defined in [30], "resembles" Hermite-Gaussian functions. Since this matrix is symmetric, one does not need an additional error removal algorithm, as described before, to orthogonalize the eigenvectors. In Chapter 4 of this work, we justify the claims of Pei, which were mainly based on numerical experiments, by showing why the eigenvectors of this matrix resemble Hermite-Gaussians and determine which eigenvector corresponds to which Hermite-Gaussian function. In the same chapter, we also resolve an ambiguity appearing in Pei's papers, the ambiguity of finding eigenvectors, when matrix dimension is a multiple of 4. In Chapter 5, we will generate a sequence of matrices generating finer approximations to Hermite-Gaussians.

In this section, we will also overview some unpublished ideas on the computation and discretization of the FrFT. Our aim, in including these works, is providing different view points for the discretization problem, which can be useful in certain application or motivating some other works.

Computation of FrFT for band-limited signals: As in [25], we assume that the signal to be transformed has Wigner distribution lying mainly in the circle of diameter R. It is easy to see that the ath domain representation of this signal is also limited to R for all a. Since the computation of FrFT of a signal, via integral kernel definition of FrFT, requires excessive sampling due to wide-band nature of the chirps, we propose that the kernel can be smoothed for this class of band-limited signals. Since signal to be transformed is band-limited to R, one can filter the signal, both at the input and at the output, by an ideal low pass filter (with a sufficiently high passband) without affecting the result. When pre and post filtering operations are accomplished, one reaches the kernel of the FrFT for band-limited signals

$$\hat{K}(t_a, t) = \int\int T(t_a, t'_a) K(t'_a, t') T(t', t) dt' dt'_a \tag{1.2}$$

where $K(t_a, t)$ denotes FrFT kernel as given in (1.1) and $T(t, t')$ denotes ideal low pass filters $T(t, t') = \frac{\sin(\pi R_x(t-t'))}{\pi(t-t')}$ where $R_x > R$.

When one inserts the spectral expansion of the kernel (see Chapter 2) in the above relation, one immediately gets the spectral of expansion of $\hat{K}$, which is given as $\hat{K}(t_a, t) = \sum_k \exp(-j\frac{\pi}{2}ka)\hat{\psi}_k(t_a)\hat{\psi}_k(t)$. The function $\hat{\psi}_k(t)$ is the low-pass filtered versions of $\psi_k(t)$. Thus FrFT matrix can be constructed by sampling these low-pass filtered Hermite-Gaussian functions and inserting them in the spectral expansion. It would be interesting to compare these Hermite-Gaussian vectors with those obtained in Chapters 4 and 5.

Additionally it is not difficult to see that 2-D convolution in (1.2) can be calculated by finding 2-D Fourier Transform of the kernel $K(t_a, t)$ and then truncating the expression in frequency-domain and finally taking the inverse Fourier Transform of the truncated function. With this method, one can express the kernel of $\hat{K}(t_a, t)$ in terms of Fresnel integrals. This calculation is facilitated by the fact that the 2-D Fourier transform of $K(t_a, t)$ with respect to the variables t_a, t has a form similar to $K(t_a, t)$ itself, a fact which is of considerable interest in itself. Finally FrFT matrix can be obtained by sampling the low pass kernel thus obtained.

Discrete FrFT by sampling the kernel of FrFT: If one compares the kernels of DFT and Fourier Transform, one may think that DFT follows from writing Riemann sum for the Fourier integral and truncating the summation to the range $n = 0 \dots N-1$. If we approximate the integral of FrFT with the same method, we get the following

$$B_a[n, k] = \frac{1}{\sqrt{N}} e^{i\pi(\cot(\phi)\frac{n^2}{N} - 2\csc(\phi)\frac{nk}{N} + \cot(\phi)\frac{k^2}{N})}$$

It is clear that B_a matrix reduces to DFT at $a = 1$, but for other values of a matrix defined is neither unitary nor satisfies index additivity. On the other hand, it is clear that B_a matrix is an approximation to continuous FrFT operator.

FrFT of sampled and replicated functions: From Poisson's theorem (see appendix), we know that one can define DFT of the samples of a function in terms of Fourier Transform of that function. By replicating a function in time and then sampling, we can generate a discrete periodic sequence in time. If we take the continuous Fourier Transform of that sequence, we get a sequence which is also periodic and discrete. If we attempt to do the same thing for FrFT, that is, examine the relation between sampled periodic sequence and its FrFT, we see that FrFT of a discrete periodic sequence is a non-periodic and continuous function in general. At the special case of $a = 1$, the signal in two domains becomes periodic and discrete and the relation between the elements in a period of two representations is DFT. But in general, there does not exist periodic and discrete sequences at the fractional orders, leading to non-existence of a matrix mapping between values in two domains (see also [31]).

Fractional Fourier Series: Ordinary Fourier series is defined as $f(t) = \frac{1}{W}\sum_k f_1(\frac{n}{W})e^{j2\pi\frac{k}{W}t}$. Following from [39], we can define DFT from Fourier series by sampling Fourier series relation at N points in a period W, that is

$$f\left(\frac{nW}{N}\right) = \frac{1}{W}\sum_{k=-\infty}^{\infty} f_1\left(\frac{n}{W}\right)e^{j2\pi\frac{kn}{N}}$$

If we evaluate the summation above with a different grouping of $k = Nq + r$, we get

$$\begin{aligned} f\left(\frac{nW}{N}\right) &= \frac{1}{W}\sum_{q=-\infty}^{\infty}\sum_{r=0}^{N-1} f_1\left(\frac{Nq+r}{W}\right)e^{j2\pi\frac{(Nq+r)n}{N}} \\ &= \frac{1}{W}\sum_{r=0}^{N-1}\left(\sum_q f_1\left(\frac{Nq+r}{W}\right)\right)e^{j2\pi\frac{rn}{N}} \end{aligned}$$

The relation between two sequences is nothing but the DFT. Following from the above discussion, one may try to define fractional Fourier series and then derive the discrete transform from the series. Fractional Fourier series can be easily generalized as

$$f(t) = \frac{A_{-\phi}}{W|\csc(\phi)|}e^{-j\pi t^2\cot\phi}\sum_{k=-\infty}^{\infty} f_a\left(k\frac{\sin\phi}{W}\right)e^{-j\pi\frac{k}{2W^2}\sin(2\phi)}e^{j2\pi\frac{k}{W}t}$$

Note that ordinary Fourier series replicates the function such that it becomes periodic with W. But fractional Fourier series involves chirp terms that makes the interpretation difficult for $|t| > W/2$. It can be seen that by straightforward sampling of the above relation, the summation index can not be reduced to a finite one as in DFT case (see also [31]).

1.2 Summary

In this chapter, we introduced the problem of discretization of the FrFT. Our aim in discretization is not only devising a method of approximation of FrFT from the samples of the input to the samples of its FrFT, but also a working definition for discrete signals. We have observed that the previous works did not satisfy all the requirements for the discrete FrFT, although some of the definitions were seemingly plausible, we will see that these definitions correspond to distinct fractionalizations of the Fourier transform, not to the FrFT defined by (1.1).

Chapter 2

Fractional Fourier Transforms

In this chapter we will investigate the definition of the FrFT and identify distinct definitions which can be obtained during the fractionalization process of the Fourier Transform. In the first section, some elementary facts related with unitary operators/matrices are studied. The second section examines eigenstructure of the ordinary Fourier Transform using unitary operator concept. The remaining sections examine the Fractional Fourier Transform and other distinct definitions.

2.1 Unitary Operators

Operators satisfying Parseval's relation are called unitary, they can also be viewed as "angle" preserving operators [22]. Another, but equivalent, definition for unitary operator $\mathbf{U}$ can be given as $\mathbf{U}^{-1} = \mathbf{U}^{H}$. In this section, we will review the following theorem about the eigenvectors of unitary matrices.

Theorem 1 *Any unitary matrix* $\mathbf{U}$ *has a complete, orthogonal set of eigenvectors.*

Proof of the theorem can be found in many references including [40, 41].

Lets recall that, if $\mathbf{U}$ has distinct eigenvalues there exists a single eigenvector set, apart form normalization, which is orthogonal due to the theorem. For the multiple eigenvalue case, the existence of an orthogonal eigenvector set is guaranteed by the theorem, but the uniqueness property is lost.

We also note that if an operator is not defined in finite dimensional space, it can not be represented with finite matrices, but with infinite matrices. One can also show the validity of the theorem discussed in this section for infinite dimensional matrices. As a result, all unitary operators, whether in finite or infinite space, have a complete and orthogonal eigenvector/eigenfunction set.

2.2 Eigenstructure of Fourier Transform

We know that Fourier Transform is unitary (Parseval's relation holds), assuring us the existence of a complete set of eigenvectors/functions of Fourier Transform. Uniqueness of this set will depend on the eigenvalues of the Fourier Transform. $\mathbf{F}$ in this section denotes either the continuous Fourier Transform or the DFT, following results are valid for both of them.

2.2.1 Eigenvalues of Fourier Transform

One can easily show that $\mathbf{F}^2 = \mathbf{F}\mathbf{F} = \mathbf{J}$; where $\mathbf{J}$ denotes coordinate inversion operator ($\mathbf{J}x(t) = x(-t)$) and $\mathbf{F}^4 = \mathbf{J}^2 = \mathbf{I}$, where $\mathbf{I}$ is identity operator. Now assume that e is an eigenvector/function of $\mathbf{F}$ with eigenvalue λ. If we multiply $\mathbf{F}^4 = \mathbf{I}$ from right by e, we get $\lambda^4 = 1$ implying $\lambda = \{1, -1, j, -j\}$. Therefore there are only 4 possibilities for the eigenvalues of $\mathbf{F}$, leading to infinitely many choices for eigenvector/functions. (DFT matrices with dimensions less than 4 are not included in this argument)

Reader may have noted that, $\mathbf{F}$ do not have to attain all of the 4 different values found above as eigenvalues. Since considering the fourth power of the identity operation $\mathbf{I}^4 = \mathbf{I}$ and repeating the same procedure, leads us to the same eigenvalue set for $\mathbf{I}$, but only one of the values from the set

$\lambda = \{1, -1, j, -j\}$ is the actual eigenvalue of $\mathbf{I}$, others are virtual roots of the 4th power operation. For the case of $\mathbf{F}$, we have all the 4 possible values as actual eigenvalues of $\mathbf{F}$, this can shown by finding a single eigenvector/function example for each eigenvalue.

2.2.2 Eigenvectors/functions of Fourier Transform

We have seen that there exists many choices for eigenvectors of $\mathbf{F}$. For the ease of visualization lets focus on N by N DFT matrix. We know that the eigenvectors of the unitary matrix DFT, with different eigenvalues are orthogonal. Since DFT has only 4 eigenvalues, this means that N dimensional space is divided into 4 hyper-planes, intersecting orthogonally with each other. For example when $N = 7$, eigenvalue multiplicity of DFT matrix, (see [42]), leads to the following distribution of eigenvalues $(1, -j, -1, j) \to (2, 2, 2, 1)$. This means that, for the DFT matrix size 7, $\lambda = 1$ plane is a spanned by 2 vectors, and orthogonal to the other planes. One can visualize the intersection of $\lambda = 1$ plane and say $\lambda = -j$ plane, as intersection of x-y plane and y-z plane in 3-D space. It is clear that taking DFT of any vector in a λ plane only maps the vector to another vector in the same plane.

To represent any vector in, say $\lambda = 1$ plane, we need two coordinates or two orthogonal basis vectors (as in x-y plane). It is clear that there exists infinitely many choices for choosing two orthogonal basis vectors, leading to infinitely many different choices for eigenvectors of DFT matrix.[1]

The following discussion, from [43], can also be fruitful to clarify the relation between λ planes and eigenstructure of DFT. First lets start with the definition of the projection matrices to the λ planes.

$$\begin{aligned}
\mathbf{P}_{1} &= \frac{1}{4}\{\mathbf{F}^3 + \mathbf{F}^2 + \mathbf{F} + \mathbf{I}\} \\
\mathbf{P}_{-1} &= \frac{1}{4}\{-\mathbf{F}^3 + \mathbf{F}^2 - \mathbf{F} + \mathbf{I}\} \\
\mathbf{P}_{j} &= \frac{1}{4}\{j\mathbf{F}^3 - \mathbf{F}^2 - j\mathbf{F} + \mathbf{I}\}
\end{aligned}$$

[1] See appendix for a Matlab program that generates a different orthogonal eigenvector set of DFT each time it is run.

$$\mathbf{P}_{-j} = \frac{1}{4}\{-j\mathbf{F}^3 - \mathbf{F}^2 + j\mathbf{F} + \mathbf{I}\}$$

The following facts about $\mathbf{P}_k$ matrices, can be easily justified by straightforward multiplication and addition of matrices.

1. $\mathbf{P}_k$ matrices are projection matrices ($\mathbf{P}_k^2 = \mathbf{P}_k$).
2. Projection spaces of $\mathbf{P}_k$ matrices are orthogonal. ($\mathbf{P}_k\mathbf{P}_l = \mathbf{0}$, $k \neq l$).
3. Direct sum of projection spaces is $\mathcal{R}^N$.
4. $\mathbf{P}_1 + \mathbf{P}_{-1}$ is the Even operator. Even$\{x[n]\} = \{\frac{1}{2}x[n] + x[-n]\}$
5. $\mathbf{P}_j + \mathbf{P}_{-j}$ is the Odd operator. Odd$\{x[n]\} = \frac{1}{2}\{x[n] - x[-n]\}$

We know that projection space of identity operator $\mathbf{I}$ is partitioned into two by Even and Odd operators. From the last two properties above, we see that the spaces of Even and Odd operators are also sub-divided into two by $\mathbf{P}_k$ operators. So projection spaces of $\mathbf{P}_k$ can be thought as generalized even/odd spaces.

Now, we proceed by showing that $\mathbf{P}_k$ matrices are projectors to λ planes by proving the projection spaces are invariant under $\mathbf{F}$ operation. That is if a given vector in the space of $\mathbf{P}_k$, vector remains in the same $\mathbf{P}_k$ space after the application of DFT operation. For example, DFT of an even(odd) vector is also an even(odd) vector.

To prove the invariance of space $\mathbf{P}_1$ under $\mathbf{F}$, it is sufficient to show that $\mathbf{F}\mathbf{P}_1 = \mathbf{P}_1$ which can be shown as $\mathbf{F}\mathbf{P}_1 = \mathbf{F}\frac{1}{4}\{\mathbf{F}^3 + \mathbf{F}^2 + \mathbf{F} + \mathbf{I}\} = \mathbf{P}_1$.

Reader can easily justify the following by using $\mathbf{F}^4 = \mathbf{I}$.

$$\begin{aligned}
\mathbf{F}\mathbf{P}_1 &= \mathbf{P}_1 \\
\mathbf{F}\mathbf{P}_{-1} &= -\mathbf{P}_{-1} \\
\mathbf{F}\mathbf{P}_j &= j\mathbf{P}_j \\
\mathbf{F}\mathbf{P}_{-j} &= -j\mathbf{P}_{-j}
\end{aligned} \tag{2.1}$$

We will now calculate the eigenvalue multiplicity of DFT. Remembering that projection spaces of $\mathbf{P}_k$ span $\mathcal{R}^N$, orthogonal with respect to each other and invariant under $\mathbf{F}$, we can easily see that the dimension of the range space of $\mathbf{P}_k$ should be equal to the multiplicity of $\mathbf{F}$ of the corresponding eigenvalue of that space. ($\{1, -1, j, -j\}$) But we know that $\mathbf{P}_k$ is a projection matrix, therefore its trace is equal to dimension of its range space which is also equal to the multiplicity of the eigenvalue denoted by k.

$$\begin{aligned}
\text{trace}\{\mathbf{P}_{\ 1}\} &= \frac{1}{4}\text{trace}\{\mathbf{F}^3 + \mathbf{F}^2 + \mathbf{F} + \mathbf{I}\} \\
&= \frac{1}{4}\text{trace}\{\mathbf{F}^{-1} + \mathbf{F}^2 + \mathbf{F} + \mathbf{I}\} \\
&= \frac{1}{4}\{\ \frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\cos(\frac{2\pi}{N}n^2) + \text{trace}\{\mathbf{J}\} + N\} \qquad (2.2)\\
\text{trace}\{\mathbf{P}_{-1}\} &= \frac{1}{4}\{-\frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\cos(\frac{2\pi}{N}n^2) + \text{trace}\{\mathbf{J}\} + N\} \qquad (2.3)\\
\text{trace}\{\mathbf{P}_{\ j}\} &= \frac{1}{4}\{-\frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\sin(\frac{2\pi}{N}n^2) - \text{trace}\{\mathbf{J}\} + N\} \qquad (2.4)\\
\text{trace}\{\mathbf{P}_{-j}\} &= \frac{1}{4}\{\ \frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\sin(\frac{2\pi}{N}n^2) - \text{trace}\{\mathbf{J}\} + N\} \qquad (2.5)
\end{aligned}$$

The expressions on the right hand side can be evaluated if results of the summations and trace$\{\mathbf{J}\}$ are known. While it is easy to calculate trace$\{\mathbf{J}\}$, the other unknown term, the Gaussian Sum identity, is very difficult to verify.[2]

$$\text{trace}\{\mathbf{J}\} = \begin{cases} 1 & N \text{ odd} \\ 2 & N \text{ even} \end{cases} \qquad (2.6)$$

Gaussian Sum:

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n^2} = \begin{cases} 1+i & N = 4m \\ 1 & N = 4m+1 \\ 0 & N = 4m+2 \\ i & N = 4m+3 \end{cases} \qquad (2.7)$$

Inserting values from (2.6) and (2.7) to (2.2)-(2.5), one can find the multiplicities of eigenvalues of $\mathbf{F}$ as indicated in Table 2.1.

[2]Reader can find four different proofs of the identity, given by Mertens, Kronecker, Schur and Gauss, in [44].

Table 2.1: Eigenvalue Multiplicity of DFT Matrix

N	1	-1	j	$-j$
$4m$	$m+1$	m	$m-1$	m
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	$m+1$	m	m
$4m+3$	$m+1$	$m+1$	m	$m+1$

An eigenvector set of DFT can also be derived from $\mathbf{P}_k$ matrices. It is easy to see that the columns of $\mathbf{P}_k$, which is given in Table 2.2, are the eigenvectors of $\mathbf{F}$.

Table 2.2: An Eigenvector Set of DFT

Eigenvalue	Eigenvector
1	$e_n^1[k] = \frac{2}{\sqrt{N}}\cos(\frac{2\pi}{N}nk) + \delta[k-n] + \delta[k+n]$
-1	$e_n^{-1}[k] = -\frac{2}{\sqrt{N}}\cos(\frac{2\pi}{N}nk) + \delta[k-n] + \delta[k+n]$
j	$e_n^j[k] = -\frac{2}{\sqrt{N}}\sin(\frac{2\pi}{N}nk) + \delta[k-n] - \delta[k+n]$
$-j$	$e_n^{-j}[k] = \frac{2}{\sqrt{N}}\sin(\frac{2\pi}{N}nk) + \delta[k-n] - \delta[k+n]$

2.3 Definition of the Continuous Fractional Fourier Transforms

In this section, we will see that specifying a set of eigenfunctions of Fourier Transform and the branch for the fractional power operations is sufficient to define the fractional Fourier Transform. First lets repeat the requirements that fractional Fourier Transforms should satisfy.

1. Unitarity of $\mathbf{F}^a$ for all a.

2. Additivity of the orders, $\mathbf{F}^{a_1}\mathbf{F}^{a_2} = \mathbf{F}^{a_1+a_2}$.

3. Reduction to ordinary Fourier Transform at $a = 1$.

We will start with elementary facts.

Fact1 Utilizing the first requirement, $\mathbf{F}^a$ has a complete and orthogonal set of eigenfunctions.

Fact2 Eigenfunctions of the $\mathbf{F}^a$ are eigenfunctions of the ordinary Fourier Transform. Since, if we assume the contrary, let e be the eigenfunction and λ_a be its eigenvalue. From requirement 2, the half order FrFT satisfies the relation $\mathbf{F} = \mathbf{F}^{\frac{1}{2}}\,\mathbf{F}^{\frac{1}{2}}$, multiplying this relation from right by e, we get $\mathbf{F}\,e = \lambda_a^2 e$, which contradicts with the assumption for $a = 1/2$ case and this special case can be generalized for all rational orders.

Fact 2, leads to two important observations. Firstly, fractional Fourier Transform for all rational orders has the same eigenfunctions with the well known special case, ordinary Fourier Transform. Secondly, the eigenvalues of the fractional Fourier Transform are fractional powers of $\{1, -1, j, -j\}$. Reader should note that due to the ambiguity in taking the fractional powers or roots of the complex numbers, one should assign a certain branch for each fractional power operation.

Using facts 1 and 2, we can define many different fractional Fourier transforms, that is using fact 1, we can expand any function in terms of a predetermined eigenfunctions of Fourier Transform.

$$f(t) = \sum_k c_k\, e_k(t) \tag{2.8}$$

$$c_k = \int_{-\infty}^{\infty} f(t') e_k^*(t') dt' \tag{2.9}$$

where in (2.8) completeness and in (2.9) orthogonality of fact 1 is used. Now using fact 2, we can take the fractional Fourier transform of $f(t)$, that is

$$\mathbf{F}^a\{f(t)\}(t) = \sum_k c_k\, (\lambda_k)^a\, e_k(t) \tag{2.10}$$

Inserting c_k from (2.9), we get the kernel of the fractional Fourier transform as

$$\mathbf{F}^a\{f(t)\}(t) = \int \left(\sum_k e_k(t)\, (\lambda_k)^a\, e_k^*(t') \right) f(t') dt' \tag{2.11}$$

The expression in the brackets of the equation (2.11) is the spectral expansion of the integral kernel of the fractional Fourier transform. One can easily

see that the definition through spectral expansion will always be unitary (since eigenvalues of the fractional transform is of unit magnitude) and reduce to ordinary Fourier Transform at the special cases (since functions generating kernel of the fractional transform are eigenfunctions of $\mathbf{F}$) and satisfy the additivity of orders requirement. One can check the order additivity property by defining $\mathbf{M} = \mathbf{F}^{a_1}\mathbf{F}^{a_2}$ and then examining its kernel. $\mathbf{M}$ can be defined as

$$K_M(t_M, t) = \int K_{a_1}(t_M, t') K_{a_2}(t', t) dt' \tag{2.12}$$

where K_M, K_{a_1}, K_{a_2} denote kernels of $\mathbf{M}, \mathbf{F}^{a_1}, \mathbf{F}^{a_2}$ respectively. Inserting K_{a_1}, K_{a_2} from (2.11), we get

$$\begin{aligned} K_M(t_M, t) &= \int \sum_{k_1} e_{k_1}(t_M)\, (\lambda_{k_1})^{a_1}\, e_{k_1}^*(t') \sum_{k_2} e_{k_2}(t')\, (\lambda_{k_2})^{a_2}\, e_{k_2}^*(t) dt' \\ &= \sum_{k_1, k_2} e_{k_1}(t_M)(\lambda_{k_1})^{a_1}(\lambda_{k_2})^{a_2} e_{k_2}^*(t) \int e_{k_2}(t') e_{k_1}^*(t') dt' \\ &= \sum_{k_1} e_{k_1}(t_M)(\lambda_{k_1})^{a_1+a_2} e_{k_1}^*(t) \end{aligned} \tag{2.13}$$

where orthogonality of $e_k(t)$ is used in the last step (Fact 1). One can easily see from (2.13) that $K_M = K_{a_1+a_2}$, proving the order additivity property.

One should note that kernel $K(t, t')$ is uniquely determined by spectral expansion, if the eigenfunction set of the Fourier Transform and the branch for the fractional power operations are specified. It is now easy to guess that distinct definitions arises from the usage of different eigenfunction sets and/or different branches for the fractional power operations. This multiplicity of definitions has led to many confusions during the process of discretization.

2.4 FrFT using Spectral Expansion

We have given the kernel of the FrFT in the first chapter. In this section we will specify the eigenfunction set and an eigenvalue assignment rule (assignment of branch) corresponding to this definition.

The eigenfunctions of the FrFT, chosen from infinitely many different possible sets of eigenfunctions of ordinary Fourier Transform, are Hermite-Gaussian functions, $\psi_k(t)$. These eigenfunctions form a complete, orthogonal set in $\mathcal{L}_2$ as expected. The eigenvalue of the kth Hermite-Gaussian function is $\lambda_k = e^{-j\frac{\pi}{2}k}$, one can observe that this value is one of the values from the set $\{1, -j, -1, j\}$, as expected. The eigenvalue assignment rule for the fractional powers is $\lambda_k^a = e^{-j\frac{\pi}{2}ka}$. Now, since the ambiguities in fractionalization are resolved, we can define kernel of the FrFT, using spectral expansion

$$K_a(t_a, t') = \sum_{k=0}^{\infty} \psi_k(t_a)\, e^{-j\frac{\pi}{2}ka}\, \psi_k(t') \tag{2.14}$$

Historically, FrFT is first defined by the spectral expansion,[2, 3], and then came the closed form definition of the kernel. Reader should note that our choice of resolving ambiguities defines a unique transform which is FrFT and any other choices will lead to distinct definitions. We also note that the process of singling out Hermite-Gaussian set from infinitely many different eigenfunction set possibilities is explained in the next chapter.

2.5 A Distinct Fractional Fourier Transform

We have seen that there exists, infinitely many fractional operators that generalize the Fourier Transform. In this section we will examine a distinct fractionalization of the Fourier Transform, which will lead to a unique discrete correspondent.

We know that there exists an ambiguity in taking the $1/N$th power of a number, and each different branch used for the $1/N$th power of an eigenvalue of the Fourier Transform leads to a different coefficient in spectral expansion of the kernel of the fractional transform, therefore affecting the definition of the fractional transform. In our definition of the FrFT, we have assigned $1/N$th power of the kth Hermite-Gaussian as $(e^{-j\frac{\pi k}{2}})^{\frac{1}{N}} \rightarrow e^{-j\frac{\pi k}{2N}}$. We can write the following for the eigenvalues of Fourier Transform in general,

$$\lambda_k = e^{-j\frac{\pi k}{2}} = e^{-j\frac{\pi}{2}(k+4p_k)} = e^{-j\frac{\pi}{2}GS_k} \tag{2.15}$$

where p_k is an arbitrary integer and $GS_k = k + 4p_k$. GS_k is called generating sequence in [45]. It is clear that by changing GS_k, we can land on any branch of

$\lambda_k^{1/N}$; so it is now possible to assign different branches for each fractional power operation encountered in the spectral expansion of the kernel. Our definition of FrFT follows from assigning $p_k = 0$ or $GS_k = k$ and using Hermite-Gaussian set as eigenfunction set.

We will now define a distinct definition of Fractional Fourier Transform, using again Hermite-Gaussians as eigenfunctions, but only changing generating sequence GS_k. Lets assign $p_k = -\lfloor \frac{k}{4} \rfloor$[3] , then $GS_k = (k)_4$ where $()_4$ denotes congruence of the argument in modulo 4. This will lead to $(e^{-j\frac{\pi n}{2}})^{\frac{1}{N}} \rightarrow e^{-j\frac{\pi (n)_4}{2N}}$. We will now examine the latter assignment rule and find the fractional transform attached to this rule.

One can observe that, different from FrFT, this new assignment rule, leads to only 4 different eigenvalues for all fractional orders. Therefore using the spectral expansion definition of the Fractional Fourier Transform we can write the following as the kernel of the new definition.

$$\begin{aligned} K(t_a, t) = \sum_{k=0}^{\infty} [&e^{-j\frac{\pi 0}{2}a}\psi_{4k}(t_a)\psi_{4k}(t) + e^{-j\frac{\pi 1}{2}a}\psi_{4k+1}(t_a)\psi_{4k+1}(t) + \ldots \\ &e^{-j\frac{\pi 2}{2}a}\psi_{4k+2}(t_a)\psi_{4k+2}(t) + e^{-j\frac{\pi 3}{2}a}\psi_{4k+3}(t_a)\psi_{4k+3}(t)] \end{aligned} \tag{2.16}$$

The summations involving Hermite-Gaussians make the result difficult to bring into a closed form. But one can bypass evaluation of the summations, by guessing that kernel can be realized as sum of 4 integer ordered Fourier transform kernels, that is

$$K(t_a, t) = c_0^a\mathbf{F}^0 + c_1^a\mathbf{F}^1 + c_2^a\mathbf{F}^2 + c_3^a\mathbf{F}^3 \tag{2.17}$$

where $\mathbf{F}^k$ denotes kernel of the k^{th} Fourier transform operator and c_k^a denotes the complex coefficients which depend on the parameter a.

If the relation (2.17) is correct, it must be satisfied by all Hermite-Gaussians. That is when (2.17) is multiplied by nth Hermite-Gaussian and integrated, we should get identical results on both sides. Equating both sides of (2.17), we can find the c_k^a coefficients. Using right sides of (2.16) and (2.17) and orthogonality of Hermite-Gaussians, we can write the following four equations for the first four Hermite-Gaussians, $\psi_0(t) \ldots \psi_3(t)$.

[3] $\lfloor x \rfloor$ is the integer part of x.

$$
\begin{aligned}
1 &= c_0^a + c_1^a + c_2^a + c_3^a \\
e^{-j\frac{\pi 1}{2}a} &= c_0^a + (-j)c_1^a + (-j)^2 c_2^a + (-j)^3 c_3^a \\
e^{-j\frac{\pi 2}{2}a} &= c_0^a + (-1)c_1^a + (-1)^2 c_2^a + (-1)^3 c_3^a \\
e^{-j\frac{\pi 3}{2}a} &= c_0^a + (j)c_1^a + (j)^2 c_2^a + (j)^3 c_3^a
\end{aligned} \tag{2.18}
$$

When we repeat the same process for the next Hermite-Gaussian, we again get the first equation in (2.18). One can easily convince oneself that the c_k^a coefficients found from the above equations, will satisfy (2.17) for all Hermite-Gaussians. Since Hermite-Gaussian set is complete, equating both sides of (2.17) for all Hermite-Gaussians is sufficient for equality of (2.17) being satisfied. When (2.18) is solved, we get the following coefficients.

$$
c_k^a = \frac{1}{4}\sum_{l=0}^{3} e^{-j\frac{\pi l(a-k)}{2}} \tag{2.19}
$$

This leads to a kernel in continuous time as

$$
K(t_a, t) = c_0^a \delta(t_a - t) + c_1^a e^{-j2\pi t t_a} + c_2^a \delta(t_a + t) + c_3^a e^{j2\pi t t_a} \tag{2.20}
$$

One can observe c_k^a acts as an interpolation function, interpolating fractional order transforms from integer order transforms.

As a result, we have seen that by changing the eigenvalue distribution of fractional transform, we have come up with a new transform, which is drastically different from FrFT. The new transform is unitary, reduces to Fourier Transform at the special cases, and satisfies the order additivity property, etc. but it is clear from the definition that this transform is somewhat infertile, in the sense that it only produces linear combination of the input and its Fourier Transform and reflected versions of these functions. That is, if the input is rectangle function, we get at the output a linear combination of sinc and rect. At the special case $a = 1$ rect dies out, only sinc function is left.

What is interesting about this definition is, it has unique discrete representation. Before starting with the description for the discrete case, lets summarize the main problem of discretization. To find a discrete equivalent of any continuous fractional transform, one should first identify the discrete

equivalents of the eigenfunctions and also make the same branch assignments for fractional power operations, or equivalently use the same GS_k sequence of the continuous transform. The main problem is: A method of identification for the discrete equivalents of the Hermite-Gaussians is not known. Therefore if one can not justify a reasoning for obtaining the discrete equivalent of continuous eigenfunctions, one can be defining a discrete fractional transform which is completely unrelated to the continuous transform that we are trying to discretize.

Returning back to the problem of finding the discrete equivalent of *this definition* we can see that we do not have to identify the discrete analogs of the Hermite-Gaussians. This surprising result is due to existence of only 4 eigenvalues for all orders. Examining (2.16) for the special case of $a = 1$, which is the Fourier Transform, one sees that the term inside the first summation corresponds to all eigenfunctions with the eigenvalue 1, therefore the first summation corresponds to $\lambda = 1$ hyper-plane and other summations denote the other planes. If we consider fractional orders, $a \neq 1$, we see that whole $\lambda = 1$ plane is multiplied by a constant. More precisely, the input is first projected onto 4 planes and then multiplied by some coefficients to construct the fractional Fourier output and these planes correspond to span of eigenfunctions with different eigenvalues. It is now clear that we do not need the discrete equivalent of each Hermite Gaussian, in the discrete version of this transform. We only need subspaces of $\lambda = \{1, -1, j, -j\}$ and we know that these spaces are uniquely determined.

To find the discrete equivalent of *this definition*, one only needs to compute an arbitrary orthogonal eigenvector set of DFT matrix and then evaluate the summations given below. One can easily check that all of the different eigenvector sets of DFT, say generated by the Matlab program in the appendix, leads to the same matrix.

$$
\begin{aligned}
K_a[k,n] = {} & \sum_{m=1}^{dim_1} e^{-j\frac{\pi 0}{2}a}\vec{e}_{1,m}\vec{e}_{1,m}^{\,T} + \sum_{m=1}^{dim_{-j}} e^{-j\frac{\pi 1}{2}a}\vec{e}_{-j,m}\vec{e}_{-j,m}^{\,T} + \\
& \sum_{m=1}^{dim_{-1}} e^{-j\frac{\pi 2}{2}a}\vec{e}_{-1,m}\vec{e}_{-1,m}^{\,T} + \sum_{m=1}^{dim_j} e^{-j\frac{\pi 3}{2}a}\vec{e}_{j,m}\vec{e}_{j,m}^{\,T}
\end{aligned} \tag{2.21}
$$

The constants $\vec{e}_{\lambda,m}$ and dim_λ denotes mth eigenvector and the dimension of the λ plane respectively. One should notice that although $e_{\lambda,m}$ are arbitrarily chosen among the infinite choices of eigenvector sets, the span of the arbitrary vectors uniquely determines the sub-spaces we need.

The other method of finding the fractional operator using c_k^a coefficients can also be utilized, and the method in discrete time is exactly same as the method for the continuous time, with an only difference of replacing **F** operator in (2.17) by DFT matrix. This results in the same values for the c_k^a as in (2.19), so the kernel in (2.21) can also be represented by

$$K_a[k,n] \;=\; c_0^a\delta[k-l] + c_1^a e^{-j\frac{2\pi}{N}kn} + c_2^a\delta[k+l] + c_3^a e^{j\frac{2\pi}{N}kn} \qquad (2.22)$$

The discrete fractional transform defined by above kernel, has been given in [30, 29]. In [30], this transform has been introduced as discrete fractional Fourier transform, we have seen in this section that the continuous analog of this discrete transform is not FrFT but another probable fractionalization of the Fourier Transform where the sole difference between the definitions is the utilization of different branches for the fractional power operations or GS_k. One can refer to [45, 46, 47] for further discussion of distinct definitions.

2.6 FrFT and other distinct definitions

We have seen that there exists infinitely many possible definitions for fractional Fourier Transforms. First of all we note that multiple definitions in fractionalization should be expected, since there exists even a multiplicity in finding square root of a number. Depending on the problem, we ignore one root due to a physical reasoning (such as time delay can not be negative or refractive index should be positive, etc.) and determine one of the roots as the principal root.

The definition of FrFT satisfies some properties that none of the other definitions satisfy. First of all, we know that FrFT can be defined from rotation of Wigner distribution for fractions of 90 degrees. If we adopt fractional rotation of Wigner Distribution as a requirement for the generalization of the Fourier Transform, there exists a unique fractional transform, which is FrFT, that satisfies this requirement [11, 48]. Additionally FrFT established strong

connections with many physical events such as diffraction, optical imaging, etc. due to equivalence of the FrFT kernel to the wave propagation in free space. One can find other applications of the FrFT in the introduction chapter. To our knowledge, we do not know another definition of fractional Fourier Transform, that is more useful in a certain area than FrFT.

2.7 Summary

In this chapter, we have examined FrFT and some other distinct definitions of fractional Fourier Transform. In the first section, we reviewed some elementary results about unitary operators and used these results to investigate the eigenstructure of the Fourier Transform. We have seen that there exists infinitely many possible sets for the eigenfunctions of the Fourier Transform. We have examined spectral expansion of the kernel of the Fourier Transform and conclude that by altering eigenfunction set and/or by choosing different branches for fractional powers, one can define different fractional Fourier Transforms. In the following section we have examined a distinct definition and find its discrete equivalent. We make an important observation that in order to find the discrete equivalent of FrFT or any other distinct definition; first of all, the discrete analog of the continuous eigenfunctions should be determined and furthermore the same GS_k must be used in both discrete and continuous definitions. We noted that although the GS_k of the continuous transform is known, one can not define an analogous discrete definition for FrFT, unless a correspondence between the eigenfunctions of continuous definition and eigenvectors of the discrete definition is established. In the last section, we tried to justify why we believe our definition stands out among the others.

Chapter 3

Eigenfunctions of FrFT

In this chapter, we will study how the eigenfunction set used in the definition of the FrFT can be singled out among the infinitely many different choices of eigenfunction sets. In the next chapter, we will attempt to make an analogous approach to the approach presented in this chapter to define the discrete equivalent of the Hermite-Gaussians functions. In the first section, we will examine commuting operators, in the next section we will construct Hermite-Gaussian set and chapter will conclude with some of the properties of the Hermite-Gaussians.

3.1 Commuting Operators

Two operators, $\mathbf{F}$ and $\mathbf{S}$, are said to commute if $\mathbf{FS} = \mathbf{SF}$. For example, partial derivative operators, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ commute for the functions of two variables x and y. A more illuminating example can be commutation of the linear time-invariant systems ($\mathcal{L}$) and shift operator ($\mathcal{E}$), that is $\mathcal{EL} = \mathcal{LE}$.

We will show that if two operators commute, there exists an eigenvector set common to both operators with, in general, different eigenvalues. For example since $\mathcal{L}$ and $\mathcal{E}$ commutes, eigenfunctions of $\mathcal{E}$, which are complex exponentials,

are also the eigenfunctions of linear time-invariant systems. Note that the eigenvalue of shift-right by t_0 operator corresponding to eigenfunction $e^{j\omega t}$ is $e^{-j\omega t_0}$; while eigenvalue of $\mathcal{L}$ of the same eigenfunction is the value of the Fourier Transform of the impulse response of $\mathcal{L}$ at ω. In this section, we will only examine operators in finite dimensions, but the results can be generalized to infinite matrices.

Theorem 2 *If matrices* $\mathbf{F}$ *and* $\mathbf{S}$ *commutes, there exists a common eigenvector set.*

Proof:
Case1: $\mathbf{S}$ has distinct eigenvalues, while eigenvalue distribution of $\mathbf{F}$ is arbitrary. Let $\vec{e_s}$ be a vector such that $\mathbf{S}\vec{e_s} = \lambda_s\vec{e_s}$. Since $\mathbf{FS} = \mathbf{SF}$,

$$\mathbf{FS}(\vec{e_s}) = \mathbf{SF}(\vec{e_s}) \;\rightarrow\; \lambda_s\,(\mathbf{F}\vec{e_s}) = \mathbf{S}(\mathbf{F}(\vec{e_s})) \;\rightarrow\; \mathbf{F}\vec{e_s} = \beta\,\vec{e_s} \tag{3.1}$$

can be written, since $\mathbf{F}\vec{e_s}$ is an eigenvector of $\mathbf{S}$ with eigenvalue λ_s and $\mathbf{S}$ has distinct eigenvalues.
Case2: $\mathbf{S}$ has non-distinct eigenvalues. We will show this case with a simple example which can be easily generalized. Assume that $\mathbf{S}\vec{e_{s1}} = \lambda_s\vec{e_{s1}}$, $\mathbf{S}\vec{e_{s2}} = \lambda_s\vec{e_{s2}}$ and $\vec{e_{s1}}, \vec{e_{s2}}$ are independent. For $\vec{e_{s1}}$, we can write the following as in case 1,

$$\mathbf{FS}(\vec{e_{s1}}) = \mathbf{SF}(\vec{e_{s1}}) \;\rightarrow\; \lambda_s\,(\mathbf{F}\vec{e_{s1}}) = \mathbf{S}(\mathbf{F}(\vec{e_{s1}})) \;\rightarrow\; \mathbf{F}\vec{e_{s1}} = c_1\,\vec{e_{s1}} + c_2\,\vec{e_{s2}} \tag{3.2}$$

where c_1, c_2 are constants, if we repeat same operation for $\vec{e_{s2}}$, we get

$$\mathbf{FS}(\vec{e_{s2}}) = \mathbf{SF}(\vec{e_{s2}}) \;\rightarrow\; \lambda_s\,(F\vec{e_{s2}}) = \mathbf{S}(F(\vec{e_{s2}})) \;\rightarrow\; \mathbf{F}\vec{e_{s2}} = c_3\,\vec{e_{s1}} + c_4\,\vec{e_{s2}} \tag{3.3}$$

We will show that by combining $\vec{e_{s1}}$ and $\vec{e_{s2}}$, we can generate the set of common eigenvectors of $\mathbf{F}$ and $\mathbf{S}$ for the eigenvalue λ_s. That is, assume that

$$\vec{e_f} = x_1\vec{e_{s1}} + x_2\vec{e_{s2}} \tag{3.4}$$

where x_1, x_2 are constants chosen such that $\mathbf{F}\vec{e_f} = \beta\,\vec{e_f}$. Rewriting $\mathbf{F}\vec{e_f}$ using (3.2) and (3.3), we get

$$\mathbf{F}\vec{e_f} \;=\; (x_1c_1 + x_2c_3)\vec{e_{s1}} + (x_1c_2 + x_2c_4)\vec{e_{s2}} \tag{3.5}$$

Since $\mathbf{F}\vec{e_f} = \beta\,\vec{e_f}$,

$$\beta = \frac{(x_1c_1 + x_2c_3)}{x_1} = \frac{(x_1c_2 + x_2c_4)}{x_2} \tag{3.6}$$

If we let $x_2 = 1$, we get the following quadratic equation for x_1

$$c_2x_1^2 + (c_4 - c_1)x_1 - c_3 \;=\; 0 \tag{3.7}$$

Therefore by solving (3.7), we can find x_1, x_2 such that $\mathbf{F}\vec{e_f} = \beta\,\vec{e_f}$. As a result two values determined from (3.7) will determine two independent vectors which are eigenvectors of both $\mathbf{S}$ and $\mathbf{F}$. If the eigenvalue multiplicity is higher than 2, the above method becomes difficult to apply, but the results will still remain valid. For a more technical proof, reader can consult [40, page 52]. ■

3.2 Hermite-Gaussians as Eigenfunctions of Fourier Transform

In this section, we will use dummy variable t in both time and frequency domains, that is $\mathbf{F}\{f(t)\}(t) = \int f(t')e^{-jtt'}dt'$.[1] Operator $\mathbf{D}$ denote differentiation in time domain (or multiplication by (jf) in frequency domain). Lets define an operator $\mathbf{S}$ as:

$$\mathbf{S} \;=\; \mathbf{D}^2 + \mathbf{F}\,\mathbf{D}^2\,\mathbf{F}^{-1} \tag{3.8}$$

where $\mathbf{F}\,\mathbf{D}^2\,\mathbf{F}^{-1}$ is the equivalent operator of $\mathbf{D}^2$ in frequency domain (multiplication by $(jf)^2$). We can express $\mathbf{FS}$ as

$$\mathbf{FS} = \mathbf{FD}^2 + \mathbf{F}^2\,\mathbf{D}^2\,\mathbf{F}^{-1} = \mathbf{FD}^2 + \mathbf{F}^2\,\mathbf{D}^2\,\mathbf{F}^{-2}\,\mathbf{F} = \mathbf{FD}^2 + \mathbf{D}^2\,\mathbf{F} = \mathbf{SF} \tag{3.9}$$

since $\mathbf{F}^2 = \mathbf{F}^{-2} = \mathbf{J}$, where $\mathbf{J}x(t) = x(-t)$, leads to $\mathbf{F}^2\,\mathbf{D}^2\,\mathbf{F}^{-2} = \mathbf{J}\mathbf{D}^2\mathbf{J} = \mathbf{D}^2$

Since $\mathbf{F}$ and $\mathbf{S}$ commute, using the results of the last section we can say that there exists a common eigenfunction set between operators $\mathbf{S}$ and $\mathbf{F}$. $\mathbf{S}$ can be expressed in time domain as

$$\mathbf{S} \;=\; \frac{d^2}{dt^2} - t^2 \tag{3.10}$$

[1] One can get the definition of Fourier Transform, given in chapter 1, by scaling time and frequency variables by $\sqrt{2\pi}$.

If we write the eigenvalue equation for $\mathbf{S}$, $\mathbf{S}f(t) = \lambda f(t)$, we obtain

$$\frac{d^2\,f(t)}{dt^2} - (\lambda + t^2)f(t) \;=\; 0 \tag{3.11}$$

By substituting $f(t) = e^{-\frac{t^2}{2}}H(t)$ in (3.11), we get

$$\frac{d^2\,H(t)}{dt^2} - 2t\frac{d\,H(t)}{dt} + \lambda_h f(t) \;=\; 0 \tag{3.12}$$

where $\lambda_h = -(\lambda + 1)$.

The differential equation has two solutions for each value of λ_h, but whatever the solutions are, we are only interested in the solutions such that $f(t) = e^{-\frac{t^2}{2}}H(t)$ remains in $\mathcal{L}_2$. Since Fourier Transform is a mapping from $\mathcal{L}_2$ to $\mathcal{L}_2$, solutions of (3.12) leading to unbounded $f(t)$ can not be eigenfunctions of $\mathbf{F}$.

Lets try to find solutions of (3.12) by power series method, that is assume that a solution of (3.12) exists in the form,

$$H(t) \;=\; \sum_{n=0}^{\infty} a_n t^n \tag{3.13}$$

Substituting $H(t)$ to (3.12), we get

$$\sum_{n=0}^{\infty} \{(n+2)(n+1)a_{n+2} - (2n - \lambda_h)a_n\}t^n \;=\; 0 \tag{3.14}$$

Coefficients of the summation, must vanish for both sides of (3.14) to be identical, this leads to

$$a_{n+2} \;=\; \frac{2n - \lambda_h}{(n+2)(n+1)}a_n \tag{3.15}$$

One can see that two solutions of (3.12), for all values of λ_h, can be given recursively from a_0 and a_1. Lets first assume that $\lambda_h = 2n$, if $\lambda_h = 0$, then two independent solutions are:

$$\begin{aligned} S_1(t) &= 1 \\ S_2(t) &= t + \frac{1}{3}t^3 + \frac{1}{10}t^5 + \frac{1}{42}t^7 + \ldots \end{aligned} \tag{3.16}$$

where in the first solution $a_0 = 1, a_1 = 0$ and in the second one $a_0 = 0, a_1 = 1$.

If $\lambda_h = 2$, then two independent solutions are:

$$\begin{aligned} S_1(t) &= 1 - t^2 - \frac{1}{6}t^4 - \frac{1}{30}t^4 + \ldots \\ S_2(t) &= t \end{aligned} \tag{3.17}$$

As a result, one can confirm that the solution of (3.12) consists of a nth degree polynomial and an infinite polynomial when $\lambda_h = 2n$. It can also be seen from the recursion formula that if $\lambda_h \neq 2n$, then both solutions of (3.12) are infinite degree polynomials. It is known from [49, page 337] that infinite degree polynomial solutions of (3.12) tends to infinity as $t \to \infty$. Therefore the only finite energy solutions of (3.11) are the finite degree polynomial solutions of $\lambda_h = 2n$ case. These polynomials are called Hermite polynomials and the generating differential equation is called Hermite equation [50]. First few Hermite polynomials are given in Table 3.1.

H_0	1
H_1	$2t$
H_2	$4t^2 - 2$
H_3	$8t^3 - 12t$
H_4	$16t^4 - 48t^2 + 12$
H_5	$32t^5 - 160t^3 + 120t$

Table 3.1: First 6 Hermite Polynomials

The function $f(t) = e^{-\frac{t^2}{2}} H_n(t)$ is called nth Hermite-Gaussian function, $\psi_n(t)$. Some properties of Hermite-Gaussians are

1. Hermite-Gaussians are orthogonal and complete in $\mathcal{L}_2$.
2. The nth Hermite-Gaussian has n real zeros.
3. The nth Hermite-Gaussian is an eigenfunction of $\mathbf{F}$ with the eigenvalue $\lambda_n = e^{-j\frac{\pi}{2}n}$.

Other properties of Hermite-Gaussians can be found at [50]. We also note, for the future reference, the possibility of identification of the order of an Hermite-Gaussian by counting the number of zeros it posseses.

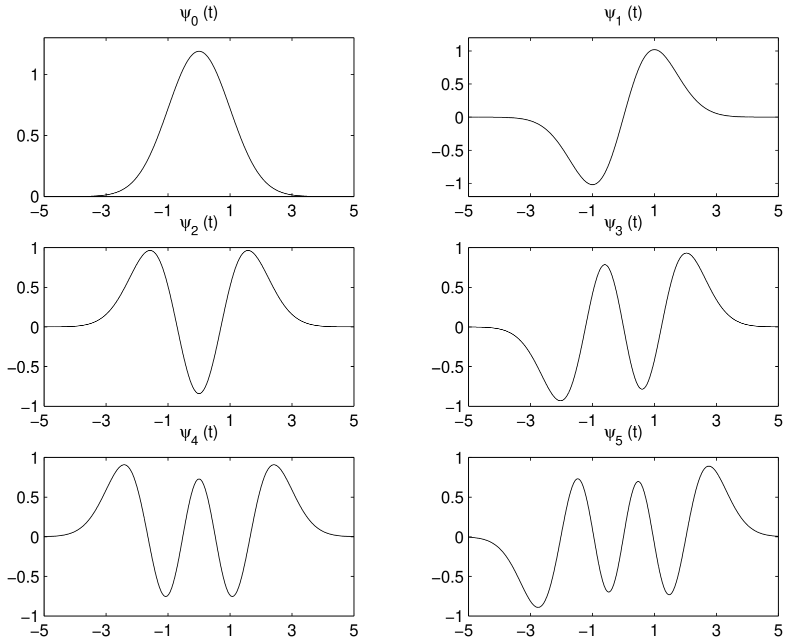


Figure 3.1: First 6 orthonormalized Hermite-Gaussian functions.

3.3 An Eigenvector Set of DFT

By the commuting $\mathbf{S}$ operator, we have shown that Hermite-Gaussians are the eigenfunctions of the continuous Fourier Transform. In this section we will try to reach an orthogonal eigenvector set of DFT that corresponds to the set of Hermite-Gaussians. If this set can be found, one can base a discrete FrFT definition on these eigenvectors, as discussed in Chapter 2.

As in the continuous case there exists infinitely many eigenvector sets of DFT, one can attempt to get the one corresponding to Hermite-Gaussians by uniformly sampling each Hermite-Gaussian. Since sampled vectors are of infinite length, one may also attempt to replicate them by period N to get periodic sequences. Using Poisson's theorem one can show that when initial sampling rate is $\frac{1}{\sqrt{N}}$, the elements of the periodic sequence in a single period, is an eigenvector of DFT. With this procedure one can get an eigenvector of DFT corresponding to each Hermite-Gaussian. It is clear that as N increases, sampled Hermite-Gaussians will be "less aliased" by the replication operation. Therefore these eigenvectors will tend to Hermite-Gaussians as size of DFT matrix increases.

Unfortunately it can be shown by numerical experiments that this eigenvector set is not orthogonal. Therefore, this set is of no use for the definition of the discrete FrFT. In the next chapter we will re-attempt to find the analogs of Hermite-Gaussians by finding the analogous operator in the discrete time to the commuting **S** operator presented in this chapter.

3.4 Summary

In this chapter, we have singled out the Hermite-Gaussian eigenfunction set of Fourier Transform, using commuting operator **S**. It is clear that there may be other commuting operators leading to different eigenfunction sets, but for the definition of FrFT given in chapter 1, we will study the Hermite-Gaussian set and the commuting operator **S**.

Chapter 4

Eigenvectors of discrete FrFT

We have seen in the previous chapters that there exists many distinct definitions for fractional Fourier transforms. In the following chapters, we will propose a discrete definition for FrFT based on discrete analogs of Hermite-Gaussians. The definition presented will satisfy the requirements of unitarity, angular additivity, reduction to DFT and the correspondence with the FrFT.

By the correspondence with the FrFT, we mean that as the dimension of the discrete transform matrix increases, the discrete transform should approach to the continuous FRFT, as in the case of DFT and Fourier transform.

We have seen in Chapter 2 that the unitary operator requirement leads the kernel of the discrete transform to be expressed by spectral expansion and the reduction to DFT requirement forces the eigenvectors used in the spectral expansion to be an eigenvector set of DFT. The reader can refer to Chapter 2, for the discussion of these comments.

In continuous FrFT, we singled out an eigenfunction set of Fourier transform, using commuting operator **S**. In this chapter we will define an operator commuting with DFT and determine the eigenvectors of this discrete commuting operator. We will attempt to find the discrete equivalents of Hermite-Gaussians, by constructing the discrete analog of the continuous commuting operator and later we will extract eigenvectors of this operator corresponding to each Hermite-Gaussian. Once eigenvectors are found, one only needs to define the branch of each fractional power operation to complete the definition by spectral expansion. It is clear that we will use the same branches that are utilized in the continuous FrFT, in order not to conflict with the correspondence requirement. As we have seen that any diversion from the correspondence with FrFT leads to distinct definitions for the discrete fractional Fourier transforms.

To summarize, we will define the discrete transform by using the eigenvectors of DFT that correspond to the first N Hermite-Gaussians. We will justify that discrete definition tends to the continuous FrFT, by showing each eigenvector of discrete commuting operator tends to its corresponding Hermite-Gaussian as matrix size $N \to \infty$.

4.1 S Matrix

In this section, we will present an operator in discrete time in analogy with the continuous **S** operator discussed in Chapter 3. In the following section, we will see the derivation of the discrete operator from the continuous one. For the purposes of this section, it is sufficient to note the similarity of the discrete time operator with **S**.

First lets introduce discrete commuting operator $\overline{\mathbf{S}}$. Let $\overline{\mathbf{S}}$ be defined by

$$\overline{\mathbf{S}} = \delta^2 + \mathbf{F}\,\delta^2\,\mathbf{F}^{-1} \tag{4.1}$$

where $\mathbf{F}$ denotes DFT matrix and δ^2denotes a second difference operator, that is $\delta^2 f_k = f_{k+1} - 2f_k + f_{k-1}$. It should be noted that the indices of the differencing operator δ^2 are cyclic. Lets first represent the operator $\mathbf{F}\,\delta^2\,\mathbf{F}^{-1}$. Since shift of $x[k] \to x[k-1]$ in time-domain is multiplication by $e^{-j\frac{2\pi}{N}k}$ in frequency-domain, one can express $\mathbf{F}\,\delta^2\,\mathbf{F}^{-1}$ as,

$$\mathbf{F}\,\delta^2\,\mathbf{F}^{-1} = e^{j\frac{2\pi}{N}k} - 2 + e^{-j\frac{2\pi}{N}k} = 2\left(\cos(\frac{2\pi}{N}k) - 1\right) \tag{4.2}$$

If we write the eigenvalue equation for $\overline{\mathbf{S}}$, $\overline{\mathbf{S}}x[k] = \overline{\lambda}x[k]$, we get

$$\left[\delta^2 + 2\left(\cos(\frac{2\pi}{N}k) - 1\right)\right] x[k] = \overline{\lambda}x[k] \tag{4.3}$$

We will now show that $\mathbf{F}$ and $\overline{\mathbf{S}}$ commutes. If we express $\mathbf{F}\overline{\mathbf{S}}$, we get

$$\mathbf{F}\overline{\mathbf{S}} = \mathbf{F}\delta^2 + \mathbf{F}^2\,\delta^2\,\mathbf{F}^{-1} = \mathbf{F}\delta^2 + \mathbf{F}^2\,\delta^2\,\mathbf{F}^{-2}\,\mathbf{F} = \mathbf{F}\delta^2 + \delta^2\,\mathbf{F} = \overline{\mathbf{S}}\mathbf{F} \tag{4.4}$$

since, $\mathbf{F}^2\,\delta^2\,\mathbf{F}^{-2} = \mathbf{J}\,\delta^2\,\mathbf{J}$ where $\mathbf{J}x[k] = x[-k]$ leads to $\mathbf{J}\,\delta^2\,\mathbf{J} = \delta^2$.

$$\begin{aligned}\mathbf{J}\left\{\delta^2\,\mathbf{J}x[k]\right\} &= \mathbf{J}\{x[-(k+1)] - 2x[-k] + x[-(k-1)]\} \\ &= x[-(-k+1)] - 2x[k] + x[-(-k-1)] = \delta^2\,x[k]\end{aligned} \tag{4.5}$$

Since $\mathbf{F}$ and $\overline{\mathbf{S}}$ commute, using our knowledge on commuting operators, we can say that there exists a common eigenvector set between $\mathbf{F}$ and $\overline{\mathbf{S}}$. One can express the eigenvalue equation (4.3) explicitly as,

$$x[k+1] + 2\cos(\frac{2\pi}{N}k)x[k] + x[k-1] = \lambda x[k] \tag{4.6}$$

where $\lambda = \overline{\lambda} + 4$. Noting the shifts in (4.6) are cyclic, we can write N equations by inserting $0 \le k \le N-1$ in (4.6).

$$\begin{bmatrix} 2 & 1 & 0 & \dots & 0 & 1 \\ 1 & 2\cos(\frac{2\pi}{N}) & 1 & \dots & 0 & 0 \\ 0 & 1 & 2\cos(\frac{2\pi}{N}2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 1 & 2\cos(\frac{2\pi}{N}(N-1)) \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{bmatrix} \tag{4.7}$$

In the first and last equations periodicity of the eigenvectors or cyclic shifts is used. We denote the matrix on the left hand side of (4.7) as **S** matrix. One can see that **S** is a tri-diagonal matrix except the ones at the first row, last column and at the last row, first column. We also note that **S** is a symmetric matrix, therefore it has an orthogonal eigenvector set. Reader may observe that $\overline{\mathbf{S}} = \mathbf{S} - 4\mathbf{I}$, therefore **S** and $\overline{\mathbf{S}}$ operators have the same eigenvectors. We will base our analysis on **S** matrix for the sake of simplicity and to be consisted with the literature.

4.2 S Matrix and Hermite-Gaussians

In this section, we will try to establish a relation between commuting operators in continuous and discrete time. Reader may have observed that the discrete commuting operator is formed from the continuous one by replacing continuous Fourier Transform and second derivative operators with their analogs in discrete time, which is DFT and second differencing operator. We will now provide an alternative approach to the operator substitution in order to clarify the relation between two operators. Since this section provides alternative but not different results, it may be skipped on a first reading.

We know that, one can generate Hermite-Gaussians from the differential equation

$$\frac{d^2 f(t)}{dt^2} - t^2 f(t) = (2n+1)f(t) \tag{4.8}$$

Solutions of (4.8), are the eigenfunctions of the Fourier transform with the definition $\mathbf{F}\{f(t)\} = \int f(t')e^{-jtt'}dt'$. By scaling time axis of Hermite-Gaussians by $\sqrt{2\pi}$, we can modify (4.8) to reveal the eigenfunctions of Fourier transform for the definition at the $a = 1$ special case of FrFT, which is $\mathbf{F}\{f(t)\} = \int f(t')e^{-j2\pi tt'}dt'$. The generating differential equation of the scaled Hermite-Gaussians, $\psi(\sqrt{2\pi}t)$, is

$$\frac{d^2 f(t)}{dt^2} - 4\pi^2 t^2 f(t) = 2\pi(2n+1)f(t) \tag{4.9}$$

Solutions of (4.9) are again orthogonal and complete. In this section we will establish a difference equation from (4.9) such that its eigenvectors form a complete orthogonal set in N dimensional space and approximate Hermite-Gaussians.

We first note that the second central difference operator is an approximation to the second derivative, since

$$\begin{aligned} \frac{\delta^2 f(t)}{h^2} &= \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} \qquad (4.10)\\ &= \frac{1}{h^2}\Bigg[\left\{ f(t) + hf'(t) + \frac{h^2}{2}f''(t) + \frac{h^3}{3!}f'''(t) + O(h^4) \right\} - 2f(t) + \ldots \\ &\quad \left\{ f(t) - hf'(t) + \frac{h^2}{2}f''(t) - \frac{h^3}{3!}f'''(t) + O(h^4) \right\}\Bigg] \\ &= f''(t) + O(h^2) \qquad (4.11) \end{aligned}$$

Discretizing the time-variable in (4.9) by taking samples h units apart from each other and replacing the second derivative by second difference, we get the following

$$\left[\frac{\delta^2}{h^2} - 4\pi^2(t_o + kh)^2\right] f(t_o + kh) = 2\pi(2n+1)f(t_o+kh) \tag{4.12}$$

Rewriting (4.12) with the notation $f_k = f(t_o + kh)$ we get

$$f_{k+1} = \left(2 + h^2 \left[\, 4\pi^2(t_o+kh)^2 + 2\pi(2n+1)\right]\right) f_k - f_{k-1} \tag{4.13}$$

It is clear that (4.13) can solved iteratively, when initial conditions are given. Solutions that are found from (4.13), will be close approximations to Hermite-Gaussians, provided that h is sufficiently small.

We know that the spectrum of the continuous Fourier transform, is aperiodic and continuous. The spectrum of DTFT is also continuous but it is periodic with 2π. DFT as an approximation to DTFT, has a spectrum that is periodic and discrete. We have seen that Hermite-Gaussians are solutions of a differential equation, that generates continuous and aperiodic functions, matching the spectrum of the continuous Fourier transform. To match the spectrum of DFT, we have written a difference equation and furthermore this difference equation should generate periodic sequences with period N. But the difference equation being close approximation to Hermite-Gaussians forces the solutions of the difference equation to be aperiodic. As a result, we will have to modify (4.12), to get periodic sequences as solutions.[1]

Lets take $t_o = 0$ in (4.12), then (4.12) becomes

$$\left[\delta^2 - 4\pi^2 h^4 k^2\right] f_k = \lambda f_k \tag{4.14}$$

Note that $2\cos(2\pi h^2 k) = 1 - 4\pi^2 h^4 k^2 + O(h^8)$, therefore replacing the quadratic term by a cosine term, in order to impose periodicity, we obtain

$$\left[\delta^2 + 2(\cos(2\pi h^2 k) - 1)\right] f_k = \lambda f_k \tag{4.15}$$

which is still an $O(h^2)$ approximation of (4.9). By fixing $h = \frac{1}{\sqrt{N}}$, we reach the following difference equation which has periodic coefficients with period N.

$$\left[\delta^2 + 2(\cos(\frac{2\pi}{N}k) - 1)\right] f_k = \lambda f_k \tag{4.16}$$

[1] One can quickly see the periodicity of eigenvectors of DFT from the Poisson's theorem.

Since coefficients of the difference equation are periodic, we are assured of the periodic solutions (see [51]).

It is now easy to see that equations (4.3) and (4.16) are identical, if one is only interested in the solutions of (4.16) which are of period N (Since shifts of the periodic solutions in (4.16) are equivalent to the cyclic shifts of (4.3)). One should note that operation $\overline{\mathbf{S}}$ in the previous section, is derived without any reference to Hermite-Gaussians, therefore without utilizing the results of this section, one does not expect to find a correspondence with Hermite-Gaussians.

4.3 Eigenvectors of S Matrix

In continuous time FrFT, commuting operator defined has distinct eigenvalues, leading to uniquely determined Hermite-Gaussian eigenfunction set. In the discrete case, when size of $\mathbf{S}$ matrix is a multiple of 4, $\mathbf{S}$ matrix has 2 eigenvalues at zero, which casts doubt on the applicability of this method, since whole aim of using commuting operators was determining eigenvectors of DFT without any ambiguity. Otherwise we could have used arbitrary eigenvector set of DFT. Fortunately, we will show that there exists a unique eigenvector set of $\mathbf{S}$ which is common to both $\mathbf{F}$ and $\mathbf{S}$, whatever the eigenvalue distribution of commuting matrix $\mathbf{S}$ is.

4.3.1 Uniqueness of Common Eigenvector Set

In [30], it has been conjectured that when N is not divisible by 4, $\mathbf{S}$ matrix has distinct eigenvalues, leading to unique eigenvectors. In this section, we will show that whatever the eigenvalue distribution of $\mathbf{S}$ is, there exists a unique set of eigenvectors common to both $\mathbf{S}$ and $\mathbf{F}$. We will start with an elementary fact.

Theorem 3 *Eigenvectors of DFT are either even or odd sequences.*

Proof: Assume contrary and let $\mathbf{F}e[k] = \lambda e[k]$, then $\mathbf{F}^2 e[k] = \lambda^2 e[k]$, but $\lambda = \{1, j, -j, -1\}$, leading to $\mathbf{J}e[k] = \pm e[k]$. This contradicts with the assumption. ■

We will define a transformation matrix $\mathbf{P}$ which decomposes vectors into even and odd components. We give the following example for vectors of length 7.

$$\mathbf{P}_7 = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.17}$$

We see from the transformation matrix $\mathbf{P}_7$ that the first 4 rows are used to calculate the even components and last 3 rows give the odd components. In general $\mathbf{P}$ matrix decomposes input to $\lfloor (N/2+1) \rfloor$ even and $\lfloor (N/2-1) \rfloor$ odd components at the output. One can notice that $\mathbf{P} = \mathbf{P}^T = \mathbf{P}^{-1}$.

Theorem 4 *Matrix* $\mathbf{PSP}^{-1}$ *is the direct sum of two tri-diagonal matrices.*

Proof: $\mathbf{S} = [r_0\, r_1\, r_2\, \ldots r_{N-1}]^T$ where r_k denotes $(k+1)$th row of $\mathbf{S}$.

$$\mathbf{PS} = \frac{1}{\sqrt{2}} \begin{bmatrix} r_0\sqrt{2} \\ r_1 + r_{N-1} \\ r_2 + r_{N-2} \\ \vdots \\ r_1 - r_{N-1} \end{bmatrix} \tag{4.18}$$

It can be seen that rows of $\mathbf{PS}$ are either even or odd sequences (examine matrix $\mathbf{S}$). By denoting the columns of $\mathbf{PS}$ as $[c_0\ c_1\ c_2\ \ldots c_{N-1}]$,

$$\mathbf{PSP}^{-1} = \frac{1}{\sqrt{2}}[c_0\sqrt{2}\ |c_1 + c_{N-1}\ |c_2 + c_{N-2}|\ \ldots\ |c_1 - c_{N-1}] \tag{4.19}$$

Since $\mathbf{PS}$ has either even or odd sequences as rows, resultant $\mathbf{PSP}^{-1}$ can be constructed from $\mathbf{PS}$ by multiplying each element of $\mathbf{PS}$ with a constant of either 0 or $\sqrt{2}$. Therefore one can observe that tri-diagonal form of $\mathbf{S}$ is preserved. As we have shown before, eigenvectors of DFT are either even or odd sequences, meaning that when eigenvectors of DFT are transformed by $\mathbf{P}$, either the first or last "half" of the entries at the output will be zero. Since we know that there exists a common eigenvector set between $\mathbf{F}$ and $\mathbf{S}$, we can conclude that $\mathbf{PSP}^{-1}$ should be written as direct sum of two matrices, that is

$$\mathbf{PSP}^{-1} = \begin{bmatrix} \mathbf{E_{vn}} & \mathbf{0} \\ \mathbf{0} & \mathbf{O_{dd}} \end{bmatrix} \tag{4.20}$$

where $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ denote tri-diagonal matrices denoting even and odd subspace matrices respectively. ■

Example 1 We will give even, odd subspace decomposition of the matrix $\mathbf{S}$ for N=6;

$$\mathbf{S} \sim \mathbf{PSP}^{-1}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \left[\begin{array}{cccc|cc} 2 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Tri-diagonal $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices can be identified from $\mathbf{PSP}^{-1}$.

It is important to note that, one can construct even or odd eigenvectors of $\mathbf{S}$ using $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices, that is assume that $\vec{e_v}$ is an eigenvector of $\mathbf{E_{vn}}$. Then zero padded $\vec{e_v}$ ($\vec{e} = [\vec{e_v}\ \ |0\ldots 0]^T$) is an eigenvector of $\mathbf{PSP}^{-1}$

in (4.20). Finally one can find eigenvectors of $\mathbf{S}$ by transforming zero padded vector through $\mathbf{P}^{-1} = \mathbf{P}$ ($\mathbf{PSP}^{-1}\vec{e} = \lambda\,\vec{e} \rightarrow \mathbf{S}(\mathbf{P}^{-1}\,\vec{e}) = \lambda\,(\mathbf{P}^{-1}\,\vec{e})$).

In the example given the vector $\vec{e} = [.77\ .60\ .16\ .04]^T$ is an eigenvector of $\mathbf{E_{vn}}$, this leads the vector

$$\mathbf{P}^{-1}\,[\vec{e}^T\ 0\ 0]^T = [.77\ .42\ .11\ .04\ .11\ .42]^T$$

to be the eigenvector of $\mathbf{S}$ with the same eigenvalue.

We have seen that we can find even, odd eigenvectors of $\mathbf{S}$ from $\mathbf{E_{vn}}$ or $\mathbf{O_{dd}}$ matrices. The next result shows that eigenvectors of $\mathbf{E_{vn}}$,$\mathbf{O_{dd}}$ matrices can be determined uniquely (apart from normalization).

Theorem 5 *Tri-diagonal matrices have distinct eigenvalues.*

Proof:

$$\mathbf{T} = \begin{bmatrix} c_0 & 1 & 0 & 0 & 0 & 0 \\ 1 & c_1 & 1 & 0 & 0 & 0 \\ 0 & 1 & c_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & c_3 & 1 & 0 \\ 0 & 0 & 0 & 1 & c_4 & 1 \\ 0 & 0 & 0 & 0 & 1 & c_5 \end{bmatrix}$$

For the ease of presentation, assume λ is an eigenvalue of $\mathbf{T}$,$|\lambda\mathbf{I} - \mathbf{T}| = 0$. If we find the determinant of the minor of the element in the first row, last column a_{16}; we get determinant as 1; therefore there exists an $(N-1)$ by $(N-1)$ submatrix of $(\lambda\mathbf{I} - \mathbf{T})$ whose determinant is non-zero, implying distinctness of eigenvalues of tri-diagonal matrices. (Taken from [52]). ■

Recapitulating the results presented so far, we have shown that it is possible to find the even/odd eigenvectors of $\mathbf{S}$ from $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices and furthermore since $\mathbf{E_{vn}}$,$\mathbf{O_{dd}}$ are tri-diagonal matrices one can uniquely determine the even/odd eigenvectors of $\mathbf{S}$ matrix through these matrices. Our motivation of defining **S** was determining an eigenvector set of DFT matrix. One can see that the unique even/odd eigenvector set of **S**must be the common eigenvector set that we are seeking for. Therefore as a result, using commuting matrix **S**, we have managed to single out a unique set of eigenvectors of DFT.

4.4 Ordering Eigenvectors of S

We have seen in the previous section that **S** generates a single set of eigenvectors of **F** without any ambiguity. In this section, we will try to order the eigenvector set of **S** in an analogy with the Hermite-Gaussian functions. Since our ultimate aim is finding a set of eigenvectors of DFT corresponding to Hermite-Gaussians, it is of vital importance to be able to identify which eigenvector of **S** corresponds to which Hermite-Gaussian.

The eigenfunctions of the continuous case, found through the continuous **S** operator, are ordered in some sense that they were named as kth Hermite-Gaussian. Reader can check from Chapter 3 that Hermite-Gaussians are ordered in terms of their number of zeros. In this section we will order the eigenvectors of **S**, by their zero-crossings.

We will first define the zero-crossing of a discrete sequence. A discrete sequence is said to have a zero crossing at k if $x[k]x[k+1] < 0$. We will order the eigenvectors of **S** in the increasing number of zero-crossings. We start with finding an explicit relation for the eigenvectors of $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices.

4.4.1 Eigenvectors of $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ Matrices

Lets first rewrite $\mathbf{E_{vn}}$ matrix,

$$\mathbf{E_{vn}} = \begin{bmatrix} c_1 & b_2 & 0 & 0 & 0 & 0 \\ b_2 & c_2 & b_3 & 0 & 0 & 0 \\ 0 & b_3 & c_3 & b_4 & 0 & 0 \\ 0 & 0 & b_4 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & b_N \\ 0 & 0 & 0 & 0 & b_N & c_N \end{bmatrix}$$

Let $p_r(\lambda)$ be the characteristic equation of the rth principal minor of $\mathbf{E_{vn}}$.

$$p_1(\lambda) = (c_1 - \lambda) \quad (4.21)$$
$$p_2(\lambda) = (c_1 - \lambda)(c_2 - \lambda) - b_2^2 \quad (4.22)$$

One can easily check that the recursion

$$p_r(\lambda) = p_{r-1}(\lambda)(c_r - \lambda) - b_r^2\, p_{r-2}(\lambda) \quad (4.23)$$

holds for $p_r(\lambda)$, where $p_0(\lambda) = 1$ for consistency.

We will now show that the eigenvector of the $\mathbf{E_{vn}}$ matrix with the eigenvalue λ is,

$$\vec{e} = [1 \quad -\frac{p_1(\lambda)}{b_2} \quad \frac{p_2(\lambda)}{b_2 b_3} \ldots (-1)^k \frac{p_k(\lambda)}{b_2 \ldots b_{k+1}} \ldots (-1)^{N-1} \frac{p_{N-1}(\lambda)}{b_2 \ldots b_N}] \quad (4.24)$$

To show that (4.24) is an eigenvector of $\mathbf{E_{vn}}$, we will explicitly write N equations of $(\mathbf{E_{vn}} - \lambda \mathbf{I})\vec{e} = 0$.

$$(c_1 - \lambda) - p_1(\lambda) = 0 \quad (4.25)$$
$$b_r(-1)^{r-2} \frac{p_{r-2}(\lambda)}{b_2 \ldots b_{r-1}} + (c_r - \lambda)(-1)^{r-1} \frac{p_{r-1}(\lambda)}{b_2 \ldots b_r} + b_{r+1}(-1)^r \frac{p_r(\lambda)}{b_2 \ldots b_{r+1}} = 0 \quad (4.26)$$
$$b_N(-1)^{N-2} \frac{p_{N-2}(\lambda)}{b_2 \ldots b_{N-1}} + (c_N - \lambda)(-1)^{N-1} \frac{p_{N-1}(\lambda)}{b_2 \ldots b_N} = 0 \quad (4.27)$$

where (4.25) and (4.27) is written from the first row and last rows of $(\mathbf{E_{vn}} - \lambda \mathbf{I})$, while (4.26) denotes the rest of the rows.

We see from (4.21) that (4.25) is satisfied and from the recursion (4.23), (4.26) is satisfied. Last equation of the eigenvalue equations, (4.27), is also satisfied since using again recursion, we get the left hand side of (4.27) as $(-1)^{N-1}\frac{p_N(\lambda)}{b_2 \ldots b_N}$, but since λ is an eigenvalue of $\mathbf{E_{vn}}$, $p_N(\lambda) = 0$. This completes the derivation for the expression for the eigenvectors of $\mathbf{E_{vn}}$. (Taken from [40, p.316])

4.4.2 Ordering Eigenvectors of $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ Matrices

In order to sort the eigenvectors using the number of zero-crossings, we need to show that eigenvectors of $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$, have distinct number of zero crossings.

To show this result, we will combine the previous explicit relation for eigenvectors and Sturm sequence property of the symmetric tri-diagonal matrices.

Theorem 6 *Let $p_r(\mu)$ denote the rth principal minor of symmetric tri-diagonal matrix evaluated for some μ. Then $s(\mu)$, the number of sign agreements of consecutive members of $p_r(\mu)$ sequence is the number of eigenvalues which are strictly greater than μ. If $p_r(\mu) = 0$, then it is assumed to have opposite sign with $p_{r-1}(\mu)$.*

Proof: One can find the proof of the theorem cited in [40, page 300]. ■

We will illustrate the theorem with an example.

Example 2

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Writing $p_r(\lambda)$ equations,

$$p_0(\lambda) = 1 \tag{4.28}$$
$$p_1(\lambda) = 1 - \lambda \tag{4.29}$$

$$p_2(\lambda) = \lambda^2 - 3\lambda + 1 \tag{4.30}$$
$$p_3(\lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda + 2 \tag{4.31}$$

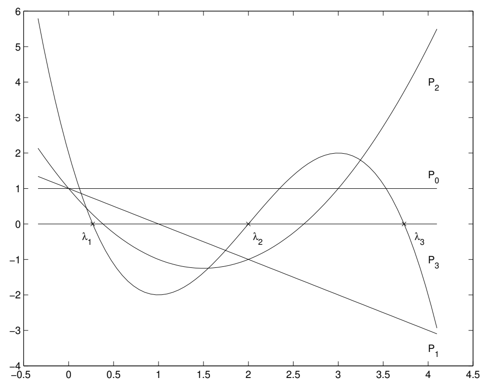


Figure 4.1: $p_r(\lambda)$ polynomials

The $p_r(\lambda)$ polynomials are shown in Figure 4.1. Eigenvalues of $\mathbf{T}$ are roots of $p_3(\lambda)$. So theorem states that the vector $p = [p_0(\mu)\ p_1(\mu)\ p_2(\mu)\ p_3(\mu)]$ has 3 sign agreements if $\mu < \lambda_1$. As μ is increased gradually, number of sign agreements decreases. This property of tri-diagonal matrices is used to find the eigenvalues of symmetric matrices. By first tri-diagonalizing symmetric matrix, and then by the bisection of the interval in which a sign change in vector p occurs, one can find an interval for eigenvalues. Accuracy of the eigenvalues can be controlled by length of the bisection interval [40, page 302].

For our case, we are interested in the number of zero-crossings of the eigenvectors, which is equivalent to the number of sign agreements of $p_r(\lambda)$ sequence where λ is an eigenvalue of of the matrix (Since $\{b_2, b_3, \ldots, b_N\}$ of $\mathbf{E_{vn}}$,$\mathbf{O_{dd}}$ are all positive). We see from the Sturm sequence theorem that the $p_r(\lambda)$ polynomials evaluated at $\lambda = \lambda_{min}$ has $(N-1)$ sign agreements. That is, the vector

$$p = [p_0(\lambda_{min})\ p_1(\lambda_{min})\ \ldots\ p_{N-1}(\lambda_{min})\ p_N(\lambda_{min})] \tag{4.32}$$

has $(N-1)$ sign agreements. Since $p_N(\lambda_{min}) = 0$ it has the opposite sign of $p_{N-1}(\lambda_{min})$ by definition, the $(N-1)$ sign agreements stays in the vector $[p_0(\lambda_{min})\ p_1(\lambda_{min})\ \ldots\ p_{N-1}(\lambda_{min})]$. One can note from (4.24) that this is equivalent to the $(N-1)$ zero crossings for the eigenvector with minimum eigenvalue. By inserting other eigenvalues, it can be seen that the number of zero-crossings of each eigenvector decreases; eventually ending with an eigenvector with no zero crossing for the $\lambda = \lambda_{max}$. As a result, in this section we have shown that $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices have eigenvectors with distinct number of zero crossings, ranging from maximum of $(N-1)$ to the minimum of 0.

Example 3
Eigenvectors of $\mathbf{E_{vn}}$ matrix of example 1 are

$$\begin{bmatrix} .78 & -.54 & .31 & .05 \\ .60 & .52 & -.58 & -.16 \\ .16 & .58 & .52 & .60 \\ .05 & .31 & .54 & -.78 \end{bmatrix}$$

where each column denotes a different eigenvector. It is clearly seen that number of zero-crossings of the eigenvectors are $\{0, 1, 2, 3\}$ respectively.

4.5 Zero Crossings of Eigenvectors of S

As discussed earlier, even eigenvectors of $\mathbf{S}$ can be found from eigenvectors of $\mathbf{E_{vn}}$, by first zero padding and then transforming with the similarity transformation $\mathbf{P}$. We will now show that even and odd eigenvectors of $\mathbf{S}$, has even and odd number of zero-crossings respectively using the results of previous section.

In the previous section we have shown that $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices have eigenvectors with distinct number of zero crossings, we will now propose a method for counting the zero-crossings of periodic sequences. Due to the

cyclic behavior of DFT; eigenvectors, $e[k]$, becomes periodic sequences when indices are extended from $0 \le k \le N-1$ to all k values. We propose that zero-crossings of the eigenvectors of DFT, should be counted in a period N including the end points of the period. That is if $e[k]$ is an N dimensional vector, $e[k] = [e_0 \ldots e_{N-1}]$, an additional zero crossing should be counted, if $e[N-1]$ and $e[N]$ have different signs or shortly $e[N-1]e[N] = e[N]e[0] < 0$. One should note that with this method of counting number of zero-crossings of periodic sequences, beginning and end of the period in which the number of zero-crossings are counted becomes unimportant, therefore number of zero-crossings of a shifted periodic sequence is always same as the original sequence. As a result the number of zero-crossings counted in this manner, becomes of a property of the periodic sequence.

Theorem 7 *Number of zero-crossings of eigenvectors of DFT, found from* $\mathbf{E_{vn}}/\mathbf{O_{dd}}$*matrices, have distinct number of zero crossings in period* $[0, N-1]$.

Proof: **Case 1:** $N = (2r+1)$. Let $\overline{e}$ be an eigenvector $\mathbf{E_{vn}}$. Then $\vec{e} = \mathbf{P}\overline{e}$ is an eigenvector of $\mathbf{S}$.

$$\vec{e} = \mathbf{P}\overline{e} = \frac{1}{\sqrt{2}}[\sqrt{2}\,\overline{e}(0)\ \ldots \overline{e}(r)\,|\,\overline{e}(r)\ \ldots\ \overline{e}(1)] \tag{4.33}$$

To count the number of zero-crossings, we pad the first element $\vec{e}(0)$ to $\vec{e}$.

$$[\vec{e}\,|\,\vec{e}(0)] = \frac{1}{\sqrt{2}}[\underbrace{\sqrt{2}\,\overline{e}(0)\ \overline{e}(1)\ \ldots\ \overline{e}(r)}_{k}\ |\ \underbrace{\overline{e}(r)\ \ldots\ \overline{e}(1)\ \sqrt{2}\,\overline{e}(0)}_{k}] \tag{4.34}$$

If $\overline{e}$ has k zero-crossings, $0 \le k \le r$, then padded vector has $2k$ zero-crossing. Repeating the same operation for $\mathbf{O_{dd}}$ eigenvectors. We get the following

$$[\vec{o}\,|\,\vec{o}(0)] = \frac{1}{\sqrt{2}}[\underbrace{0\ \overline{o}(1)\ \ldots\ \overline{o}(r)}_{k}\ |\ \underbrace{-\overline{o}(r)\ \ldots\ -\overline{o}(1)\ 0}_{k}] \tag{4.35}$$

We see that if $\overline{o}$ has k zero-crossings, $0 \le k \le r-1$, then padded $\vec{o}$ vector has $2k+1$ zero crossings. Note that zeros at two sides of the padded vectors do not introduce a zero-crossing, since sign of zero is assumed to be the opposite sign of the preceding term, by the Sturm sequence theorem.

Case 2: $N = (2r)$. Assume $\mathbf{E_{vn}}\overline{e} = \lambda\overline{e}$, padded $\vec{e}$ vector becomes

$$[\vec{e}\,|\,\vec{e}(0)] = \frac{1}{\sqrt{2}}[\underbrace{\sqrt{2}\,\overline{e}(0)\ \overline{e}(1)\ \ldots\ \overline{e}(r-1)\ |\,\overline{e}(r)|}_{k}\,\overline{e}(r-1)\ \ldots\ \overline{e}(1)\ \sqrt{2}\,\overline{e}(0)] \quad (4.36)$$

We see that padded $\vec{e}$ vector has again $2k$ zero-crossings, $0 \le k \le r$. When we repeat for odd vectors, we get

$$[\vec{o}\,|\,\vec{o}(0)] = \frac{1}{\sqrt{2}}[\underbrace{0\ \overline{o}(1)\ \ldots\ \overline{o}(r-1)|\ 0\ |}_{k}-\overline{o}(r-1)\ \ldots\ -\overline{o}(1)\ 0] \quad (4.37)$$

As a result, padded odd eigenvectors of $\mathbf{S}$, has $2k+1$ zero-crossings where $0 \le k \le r-2$. ■

With this theorem, we have showed that the eigenvectors of $\mathbf{S}$ matrix have distinct number of zero-crossings. Therefore each eigenvector can be identified uniquely by its number of zero-crossings.

4.6 Comparison of Eigenvectors of S with Hermite-Gaussians

In this section, we will numerically compare the eigenvectors of $\mathbf{S}$ and samples of Hermite-Gaussian functions.

Lets review shortly our method of finding discrete equivalents of Hermite-Gaussians. First of all, the even/odd eigenvectors of $\mathbf{S}$ are found using $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices and then these eigenvectors are sorted in the increasing number of zero-crossings. Since we have proved distinctness of eigenvalues of $\mathbf{E_{vn}}/\mathbf{O_{dd}}$ and distinct number of zero-crossings for eigenvectors, we are able to identify uniquely and sort the eigenvectors in the increasing number of zero-crossings.

We note that in order to sort the eigenvectors by their number of zero crossings, it is not needed to count the actual number of zero crossings of each vector. From the proof, we presented for the distinctness of zero crossings using Sturm Sequence Theorem, it is clear that the number of zero-crossings an eigenvector posses, can be determined by counting the number of eigenvalues which are higher than the eigenvalue of that eigenvector. More clearly, the even eigenvector of $\mathbf{S}$ with no zero-crossings is the eigenvector of $\mathbf{E_{vn}}$ with the highest eigenvalue, and the eigenvector with two zero crossings is the one with the second highest eigenvalue. Therefore with this method, we also overcome the numerical problem of counting zero-crossings. That is if the eigenvector, whose number of zero-crossings is being counted, has some elements that are very small in magnitude, the quantization errors during the calculation of this vector can flip the signs of these elements. Since these elements are almost zero, accuracy of these vectors will remain same, but the number of zero-crossing of these vectors will be wrongly detected. This problem is evidenced by MATLAB simulations of $\mathbf{S}$ matrices with $N > 70$.

We will compare the eigenvectors, with the samples of the Hermite-Gaussians sampled with $\frac{1}{\sqrt{N}}$ around zero[2] , which is the same sampling period, $h = \frac{1}{\sqrt{N}}$, that is used to generate the periodic difference equation from Hermite-Gaussian generating differential equation. For the comparison purposes, the samples of the continous Hermite-Gaussians are normalized such that they have unit norm, as the eigenvectors of the $\mathbf{S}$ matrix.

In Figure 4.2, we compare the eigenvector of 8 by 8 $\mathbf{S}$ matrix with k zero-crossings with the samples of the kth Hermite-Gaussian function. We also define the error between two vectors as the $\mathcal{L}_2$ norm of the difference between the eigenvector and N samples of the kth Hermite Gaussian around zero.

$$\text{error} \quad = \quad ||(\psi_k(\frac{n}{\sqrt{N}}) - e_k[n])||_2 \quad (4.38)$$

In Figure 4.3, we examine the case of $N = 25$, for the first 8 eigenvectors that is eigenvectors with $\{0, \ldots, 7\}$ zero crossings. One can easily the see the evolution of the eigenvectors to the Hermite-Gaussians, as N increases.

In Figure 4.4, we present the error defined above for different values of N.

[2]For all N values, samples are taken such that a sample exists at zero and "half" of the samples lie in the negative t axis, the other "half" in positive axis where the distance between two consecutive samples is $\frac{1}{\sqrt{N}}$. For N even case, two "halves" will not be equal in number of elements, but one can take extra sample either in positive or negative axis without changing the resultant vector since Hermite-Gaussians are even/odd functions.

It is seen from Figure 4.4 that as N increases the error between samples of kth Hermite-Gaussian and the corresponding eigenvector decreases, but it is also witnessed from Figure 4.4 that for all values of N, the eigenvectors with many zero-crossings are not in a good agreement with Hermite-Gaussians. These two factors, where the former favors the **S** matrix method, and the latter degrades the method, is expected; since firstly as N increases $h = \frac{1}{\sqrt{N}}$ decreases so that finite difference approximation improves and secondly if it were possible to find all eigenvectors of **S** in a very good agreement with Hermite-Gaussians, then these vectors can not be orthogonal and they can not be eigenvectors of the DFT matrix. Since it is known that the samples of Hermite-Gaussians are not orthogonal and they are not eigenvectors of DFT. As a result, exact samples of the Hermite-Gaussians should be altered if one wants to get an orthogonal eigenvector set of DFT. In the next chapter, we will find a sequence of **S** matrices such that the approximation to Hermite-Gaussians is improved for all eigenvectors, while orthogonality and eigenrelation with DFT is preserved.

4.7 Summary

In this chapter, we have found an eigenvector set of DFT via **S** matrix. We have seen that **S** matrix generates a unique eigenvector set of DFT which can be ordered by the number of zero-crossings of each eigenvector. Therefore, N by N **S** matrix generates an ordered sequence of orthogonal vectors, which can be put into one-to-one correspondence with first N Hermite-Gaussian functions. In the last section, we have compared these eigenvectors with Hermite Gaussians by numerical simulations.

4.8 Notes

S matrix is first introduced without presenting any relation with Hermite-Gaussians in [30]. Authors propose that using **S** one can find a unique orthogonal eigenvector set of DFT. They base uniqueness of eigenvectors on their

conjecture of distinctness of eigenvalues. In the rest of the paper, authors try to strengthen their conjecture for distinctness by analysis of **S** matrix. Although we did not improve the conjecture, we bypassed this requirement for distinctness by showing whatever the distribution of eigenvalues of **S**, there exists a unique common eigenvector set between **S** and DFT.

Although we were aware of [30], we were not expecting the eigenvectors found from **S** matrix to be useful, since this eigenvector set constitutes another eigenvector set of DFT, of which we can generate infinitely many (see Appendix). Recently Pei has published a letter,[37], emphasizing on the "similarity" of the eigenvectors found from **S** with Hermite-Gaussians. Pei based his observations on numerical simulations with **S** matrix. This work of Pei has initiated the work presented in this chapter and the next chapter. In his letter, Pei claimed that "DFT shifted eigenvectors of **S** with k zero-crossings is similar to the kth Hermite Gaussian". In this chapter we have not only shown the existence of an eigenvector with k zero-crossings, but also determined a method of finding that eigenvector without explicitly counting the zero-crossings. For the similarity between the eigenvectors and the Hermite-Gaussians, we have presented an analogy between the difference equation leading to **S** matrix and the Hermite-Gaussian generating differential equation. In the next chapter we will find a sequence of **S** matrices that approximates Hermite-Gaussians more closely.

Further research on **S** matrices, can be based on Mathieu equations [53] and Sturm-Liouville problems (For continuous-case [49, 54], for discrete Sturm-Liouville problem with periodic boundary conditions.[55, 56, 57]).

Figure 4.2: Hermite-Gaussians and Eigenvectors of **S** matrix, N=8.

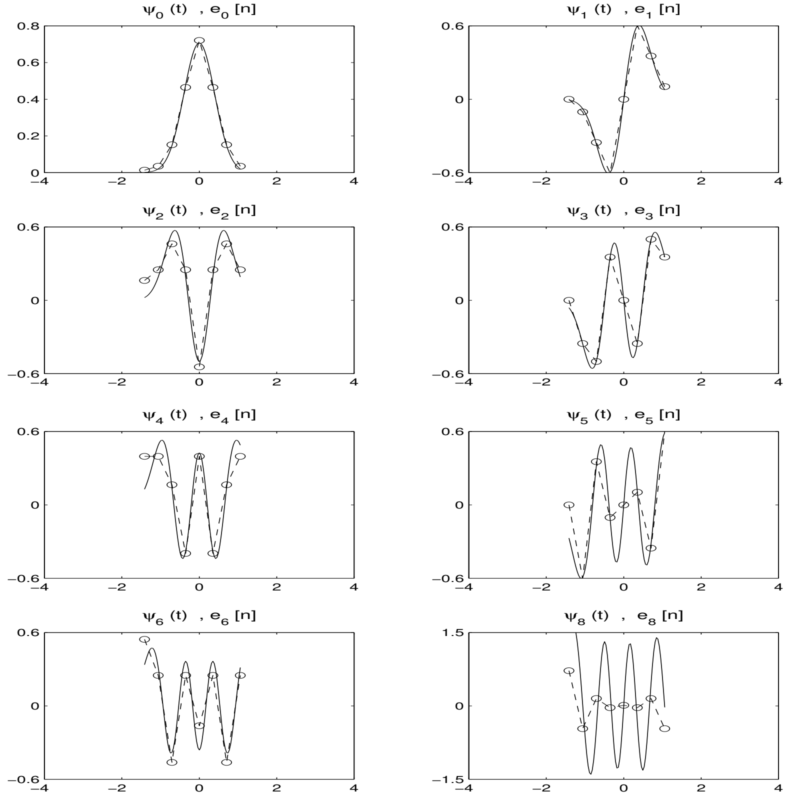


Figure 4.3: Hermite-Gaussians and Eigenvectors of **S** matrix, N=25.

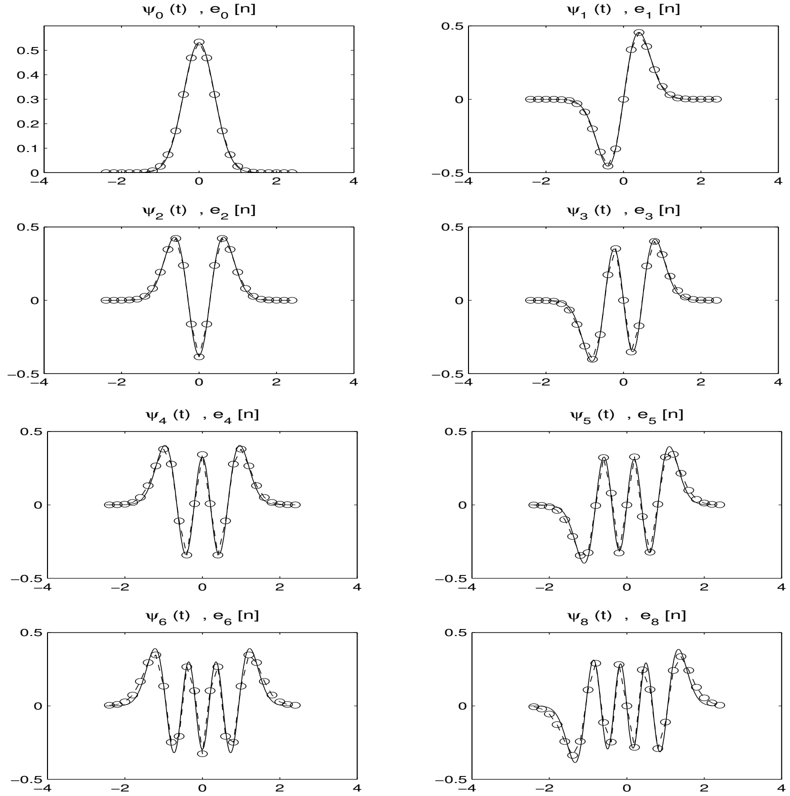
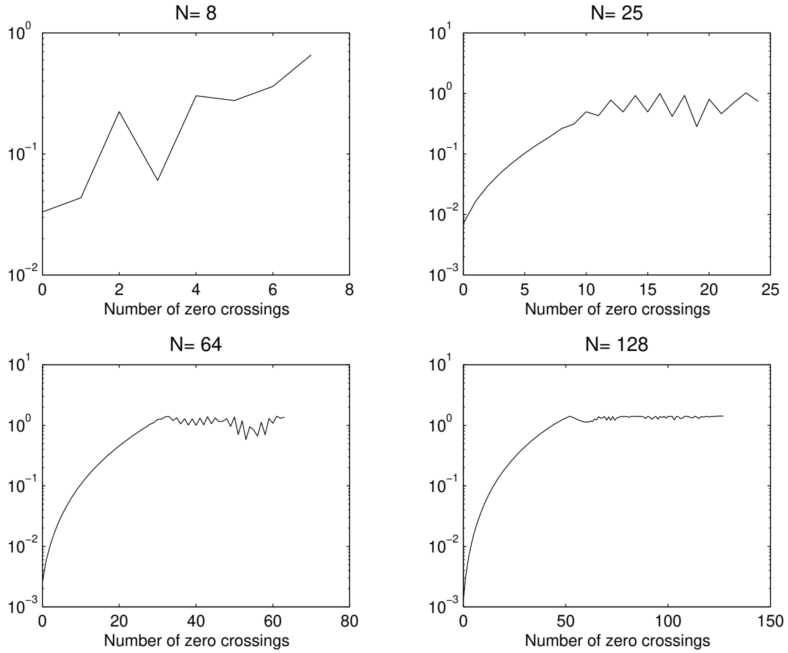


Figure 4.4: Error sequences of $\{8, 25, 64, 128\}$ dimensional S matrices.



Chapter 5

Higher Order Approximations

In this chapter, we will derive a sequence of matrices, $\mathbf{S}_{2k}$, that generates finer approximations for Hermite-Gaussians. We will first re-examine the $\mathbf{S}$ matrix defined in Chapter 4 in a broader context and then derive the finer approximation matrices. We will refer to the $\mathbf{S}$ matrix of the last chapter as $\mathbf{S}_2$ which is the crudest approximation matrix of the sequence.

5.1 $\mathbf{S}_2$ matrix

In this section, we will review the approximation of $\mathbf{S}_2$ to Hermite Gaussians. We will first establish some relations on the difference operators.[1] Lets start with the shift operator $\mathbf{E}^h$, $\mathbf{E}^h f(t) = f(t+h)$. $\mathbf{E}^h$ can also be represented as a summation using Taylor series expansion of $f(t+h)$.

$$\begin{aligned} \mathbf{E}^h &= 1 + \frac{h\mathbf{D}}{1} + \frac{h^2\mathbf{D}^2}{2!} + \cdots + \frac{h^k\mathbf{D}^k}{k!} + \cdots \\ &= e^{h\mathbf{D}} \end{aligned} \tag{5.1}$$

[1] Reader can refer to appendix for a brief discussion of finite-difference operators.

where $\mathbf{D}$ is the derivative operator. If we denote the continuous Fourier Transform by $\mathbf{F}$, $\mathbf{F}\,\mathbf{E}^h\,\mathbf{F}^{-1}$ can be expressed as

$$\begin{aligned}
\mathbf{F}\mathbf{E}^h\mathbf{F}^{-1} &= \mathbf{F}\left[1+\frac{h\mathbf{D}}{1}+\frac{h^2\mathbf{D}^2}{2!}+\cdots+\frac{h^k\mathbf{D}^k}{k!}+\cdots\right]\mathbf{F}^{-1} \\
&= 1+\frac{h(j2\pi f)}{1}+\frac{h^2(j2\pi f)^2}{2!}+\cdots+\frac{h^k(j2\pi f)^k}{k!}+\cdots \\
&= e^{j2\pi fh} \qquad (5.2)
\end{aligned}$$

which is well known shift property of the Fourier transform.

Second central difference operator, $\frac{\delta_h^2}{h^2}=\mathbf{E}^h-2+\mathbf{E}^{-h}$, is an approximation to the second derivative operation, that is

$$\frac{\delta_h^2}{h^2}=\frac{e^{h\mathbf{D}}-2+e^{-h\mathbf{D}}}{h^2}=\mathbf{D}^2+\frac{h^2}{12}\mathbf{D}^4+O(h^4) \qquad (5.3)$$

If we express $\mathbf{F}\delta_h^2\mathbf{F}^{-1}$, we get

$$\begin{aligned}
\mathbf{F}\delta_h^2\mathbf{F}^{-1}=\mathbf{F}\left[\frac{e^{h\mathbf{D}}-2+e^{-h\mathbf{D}}}{h^2}\right]\mathbf{F}^{-1} &= \frac{e^{j2\pi hf}-2+e^{-j2\pi hf}}{h^2}=\frac{2\left(\cos(2\pi hf)-1\right)}{h^2} \\
&= -4\pi^2f^2+\frac{h^2}{12}\left(16\pi^4f^4\right)+O(h^4) \qquad (5.4)
\end{aligned}$$

We have seen in the previous chapters that the generating differential equation for Hermite-Gaussians is

$$\begin{aligned}
\left(\mathbf{D}^2+\mathbf{F}\mathbf{D}^2\,\mathbf{F}^{-1}\right)f(t) &= \lambda f(t) \qquad (5.5)\\
\left(\mathbf{D}^2-4\pi^2\,t^2\right)f(t) &= \lambda f(t) \qquad (5.6)
\end{aligned}$$

where both time and frequency domain is represented by the dummy variable t in (5.5).

In this section, we will rewrite the differential equation (5.6), as a difference equation by replacing $\mathbf{D}^2$ with $\frac{\delta_h^2}{h^2}$ in (5.5), that is

$$\frac{\delta_h^2}{h^2}+\mathbf{F}\,\frac{\delta_h^2}{h^2}\,\mathbf{F}^{-1} = \frac{\delta_h^2}{h^2}+\frac{2\left(\cos(2\pi th)-1\right)}{h^2} \qquad (5.7)$$

One can also express (5.7), using (5.3) and (5.4) as,

$$\frac{\delta_h^2}{h^2}+\frac{2\left(\cos(2\pi th)-1\right)}{h^2} = \mathbf{D}^2-4\pi t^2+\frac{h^2}{12}\left(D^4+16\pi^4\,t^4\right)+O(h^4) \qquad (5.8)$$

where the approximation to (5.6) is of $O(h^2)$. If we explicitly write (5.7), we get the following

$$f(t+h)-2f(t)+f(t-h)+2\left(\cos(2\pi th)-1\right)f(t) = h^2\lambda f(t) \qquad (5.9)$$

We see from (5.8) that, the recursion defined above is an approximation on the order $O(h^2)$ of the Hermite-Gaussian generating differential equation. One can easily convince oneself that if $f(t)$ and $f(t-h)$ values are given, $f(t+kh)$ can be calculated for an arbitrary integer k. Denoting $f(t+kh)$ by f_k recursion can be rewritten as

$$f_{k+1}-2f_k+f_{k-1}+2\left(\cos(2\pi kh^2)-1\right)f_k = h^2\lambda f_k \qquad (5.10)$$

where t in (5.9) is replaced by kh, by taking $f_0=f(0)$. By using the notation $\delta^2 f_k=f_{k+1}-2f_k+f_{k-1}$, we can write (5.10) compactly as,

$$\delta^2 f_k+2\left(\cos(2\pi kh^2)\right)f_k = \lambda f_k \qquad (5.11)$$

where λ's in (5.10) and (5.11) are not generic.

As discussed earlier, if we expect to find the eigenvectors of DFT from (5.11), we should only consider periodic solutions with period N. By fixing $h=\frac{1}{\sqrt{N}}$ in (5.11), we get a difference equation with periodic coefficients and existance of periodic solutions is assured. Reader can check that when $h=\frac{1}{\sqrt{N}}$, the difference equation (5.11) is exactly the difference equation generating the $\mathbf{S}$ matrix of the last chapter.

5.2 $\mathbf{S}_{2k}$ matrices

In this section, we will define $\mathbf{S}_{2k}$ matrices such that the eigenvectors of $\mathbf{S}_{2k}$ approximate of the Hermite-Gaussians on the order of $O(h^{2k})$. We will first define a sequence of approximations to the second derivative operator denoted by $(\widetilde{\mathbf{D}^2})_{2k}$ where $2k$ is the order of approximation. The operator δ_h^2examined in the previous section, is an approximation of the order $O(h^2)$, therefore it can be named as $(\widetilde{\mathbf{D}^2})_2$ in the general setting of approximations.

Theorem 8 *$O(h^{2k})$ approximation of $\mathbf{D}^2$ can be expressed as*

$$(\widetilde{\mathbf{D}^2})_{2k} = \sum_{m=1}^{k} (-1)^{m-1} \frac{2\left[(m-1)!\right]^2}{(2m)!} \delta_h^{2m} \qquad (5.12)$$

Proof of the theorem is given in the appendix.

It can be seen that $O(h^{2k})$ approximation is exact for the polynomials of degree $2k$. Therefore as $2k \to \infty$, we get the following equality for the second derivative.

$$\mathbf{D}^2 = \delta_h^2 - \frac{1}{12}\delta_h^4 + \frac{1}{90}\delta_h^6 + \cdots \qquad (5.13)$$

To generalize the work presented in the previous section, we define the following

$$\mathbf{S}_{2k} = (\widetilde{\mathbf{D}^2})_{2k} + \mathbf{F}\,(\widetilde{\mathbf{D}^2})_{2k}\,\mathbf{F}^{-1} \qquad (5.14)$$

where $\mathbf{F}$ represents the continuous Fourier transform.

Example 1

We will examine $\mathbf{S}_4$ matrix in this example. $(\widetilde{\mathbf{D}^2})_4$ and $\mathbf{F}\,(\widetilde{\mathbf{D}^2})_4\,\mathbf{F}^{-1}$ operators can be expressed as.

$$\begin{aligned}(\widetilde{\mathbf{D}^2})_4 = \delta_h^2 - \frac{1}{12}\delta_h^4 &= \frac{1}{h^2}\left[(\mathbf{E}^h - 2 + \mathbf{E}^{-h}) - \frac{1}{12}(\mathbf{E}^h - 2 + \mathbf{E}^{-h})^2\right] \\ &= \frac{1}{h^2}\left[-\frac{1}{12}\mathbf{E}^{2h} + \frac{4}{3}\mathbf{E}^h - \frac{5}{2} + \frac{4}{3}\mathbf{E}^{-h} - \frac{1}{12}\mathbf{E}^{-2h}\right] \qquad (5.15)\end{aligned}$$

$$\mathbf{F}\,(\widetilde{\mathbf{D}^2})_4\,\mathbf{F}^{-1} = \frac{2}{h^2}\left[-\frac{1}{12}\cos\left(2\pi h^2(2k)\right) + \frac{4}{3}\cos\left(2\pi h^2(k)\right) - \frac{5}{4}\right] \qquad (5.16)$$

Then using last two relations, $\mathbf{S}_4$ can be found after fixing $h = \frac{1}{\sqrt{N}}$.

$$\delta_h^2 f_k - \frac{1}{12}\delta_h^4 f_k + 2\left(-\frac{1}{12}\cos(\frac{2\pi}{N}(2k)) + \frac{4}{3}\cos(\frac{2\pi}{N}k) - \frac{5}{4}\right) f_k = \lambda f_k \qquad (5.17)$$

Note that the resulting difference equation has periodic coefficients with period N. One can observe that periodic solutions of (5.17) is equivalent to the eigenvectors of the following matrix,

$$\mathbf{S}_4 = \begin{bmatrix} 2C_0 & \frac{4}{3} & -\frac{1}{12} & 0 & 0 & -\frac{1}{12} & \frac{4}{3} \\ \frac{4}{3} & 2C_1 & \frac{4}{3} & -\frac{1}{12} & 0 & 0 & -\frac{1}{12} \\ -\frac{1}{12} & \frac{4}{3} & 2C_2 & \frac{4}{3} & -\frac{1}{12} & 0 & 0 \\ 0 & -\frac{1}{12} & \frac{4}{3} & 2C_3 & \frac{4}{3} & -\frac{1}{12} & 0 \\ 0 & 0 & -\frac{1}{12} & \frac{4}{3} & 2C_4 & \frac{4}{3} & -\frac{1}{12} \\ -\frac{1}{12} & 0 & 0 & -\frac{1}{12} & \frac{4}{3} & 2C_5 & \frac{4}{3} \\ \frac{4}{3} & -\frac{1}{12} & 0 & 0 & -\frac{1}{12} & \frac{4}{3} & 2C_6 \end{bmatrix} \qquad (5.18)$$

where $\{C_0, C_1, \ldots\}$ denotes the sum of the cosine terms in (5.17). One can observe that $\mathbf{S}_4$ becomes a five diagonal matrix and it is clear that $\mathbf{S}_{2k}$ becomes more and more banded matrix as k increases. On the other hand, one can see that the terms on the "far" diagonals become considerably small compared with the terms on the main diagonal.

Theorem 9 *$\mathbf{S}_{2m}$ matrix commutes with DFT matrix.*

Proof: $\mathbf{S}_{2m} = (\widetilde{\mathbf{D}^2})_{2m} + \mathbf{F}\,(\widetilde{\mathbf{D}^2})_{2m}\,\mathbf{F}^{-1}$. Lets call the two terms constructing $\mathbf{S}_{2m}$ as A and B, that is

$$(\widetilde{\mathbf{D}^2})_{2m} = c_m\mathbf{E}^{mh} + c_{m-1}\mathbf{E}^{(m-1)h} + \cdots + c_0 + \cdots + c_m\mathbf{E}^{-mh} = A \qquad (5.19)$$

$$\mathbf{F}\,(\widetilde{\mathbf{D}^2})_{2m}\,\mathbf{F}^{-1} = 2\left(c_m\cos(2\pi h^2 mk) + c_{m-1}\cos(2\pi h^2(m-1)k) + \cdots + \frac{c_0}{2}\right) = B \qquad (5.20)$$

We know that the cyclic shift operation ($x[k] \to x[k-m]$) in time-domain is multiplication by $e^{-j\frac{2\pi}{N}mk}$ of the DFT representation of $x[k]$ in DFT-domain. Since we are only interested in the periodic solutions of difference equation $(\widetilde{\mathbf{D}^2})_{2m} + \mathbf{F}\,(\widetilde{\mathbf{D}^2})_{2m}\,\mathbf{F}^{-1}$, one sees that shift $\mathbf{E}^{mh}$ corresponds to cyclic shift by m units, leading to the fact that $(DFT)(A)(DFT)^{-1} = B$ when $h = \frac{1}{\sqrt{N}}$ is taken in B. On the other hand using the reverse of the same argument one can also show that $(DFT)(B)(DFT)^{-1} = (A)$. Proving that

$$(DFT)\mathbf{S}_{2m}(DFT)^{-1} = (DFT)(A+B)(DFT)^{-1} = B + A = \mathbf{S}_{2m}$$

or equivalently $(DFT)\mathbf{S}_{2m} = \mathbf{S}_{2m}(DFT)$ ■

5.3 Eigenvectors of $\mathbf{S}_{2k}$

We have shown that $\mathbf{S}_{2k}$ commutes with $\mathbf{F}$, implying the existence of the common eigenvector set between $\mathbf{S}_{2k}$ and $\mathbf{F}$. We have examined the special case of $\mathbf{S}_2$ in the previous chapter and managed to determine a unique set of common eigenvectors and also identify the Hermite-Gaussian that each eigenvector corresponds, using the number of zero crossings argument. Unfortunately the results proven in the last chapter can not be generalized for all k values, since during the process of proving, we have utilized many results from the theory of tri-diagonal matrices and $\mathbf{S}_{2k}$ matrices becomes more and more banded as k increases. Nevertheless if we think of $\mathbf{S}_2$ matrix as the main approximation matrix and the other matrices as the additional terms to "fine tune" the results derived in the last chapter, one may expect to achieve results similar to the ones of the last chapter.

In this section we will find the eigenvectors of $\mathbf{S}_{2k}$ with the method derived for $\mathbf{S}_2$. That is first $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices are constructed from $\mathbf{S}_{2k}$ and eigenvectors of $\mathbf{E_{vn}}/\mathbf{O_{dd}}$ are found, and then eigenvectors of $\mathbf{S}_{2k}$ are derived by the similarity transformation $\mathbf{P}$ of the zero padded eigenvectors of $\mathbf{E_{vn}}/\mathbf{O_{dd}}$. (See Chapter 4) Eigenvectors are sorted using the same method utilized for $\mathbf{S}_2$, that is for $\mathbf{S}_2$, we have shown that the even eigenvector with no zero-crossings is the eigenvector of $\mathbf{E_{vn}}$ which has the highest eigenvalue. We claim that the method of ordering, as discussed in Chapter 4, should be valid for all S_{2k} matrices. In the last chapter we have seen that the solutions of the crudest approximation matrix $\mathbf{S}_2$ catches the main functional behavior of Hermite-Gaussians. Therefore one may expect the validity of the same ordering method for all $\mathbf{S}_{2k}$ matrices.

In Figure 5.1, we present the comparison of the samples of Hermite-Gaussians (sampled with $h = \frac{1}{\sqrt{N}}$ as in Chapter 4) with eigenvectors found from $\mathbf{S}_2$, $\mathbf{S}_4$, $\mathbf{S}_6$ matrices where $N = 8$.

In Figure 5.2, we present a similar comparison for $N = 25$ where eigenvectors of $\mathbf{S}_2$, $\mathbf{S}_8$, $\mathbf{S}_{24}$ matrices are compared.

In Figure 5.3, we present the error between Hermite-Gaussians and eigenvectors of $\mathbf{S}_{2k}$ as k increases for the cases $N = 8, 16, 64, 128$.

One can check the validity of the method for the ordering of the eigenvectors of $\mathbf{S}_{2k}$ matrices from Figure 5.3. If the method of detecting the order of Hermite-Gaussians fails for some reason, then the error curves in Figure 5.3 should not stay completely under each other as k increases. At least a single eigenvector with increasing error (as k increases) would have implied the discrepancy of the method, but until now we have not observed such a situation.

5.4 Comparison of the Hermite-Gaussians and the Eigenvectors of $\mathbf{S}_{2k}$

In this section we will compare the properties of the eigenvector set found from $\mathbf{S}_{2k}$ with Hermite-Gaussians. First we will list some properties of Hermite-Gaussians.

1. Hermite-Gaussians satisfy a generating differential equation.
2. Hermite-Gaussians are eigenfunctions of continuous Fourier Transform .
3. Hermite-Gaussians form a complete orthogonal set in $\mathcal{L}_2$.
4. Hermite-Gaussians can be sorted by their number of zeros.
5. The kth Hermite-Gaussian with k zeros has the eigenvalue of $e^{-j\frac{\pi k}{2}}$ under Fourier Transform operation.

If we list corresponding properties for eigenvectors of $\mathbf{S}_{2k}$.

1. Eigenvectors satisfy a generating difference equation which is an approximation to Hermite-Gaussian generating differential equation.
2. Eigenvectors of are eigenfunctions of DFT.
3. Eigenvectors form a complete and orthogonal set in $\mathcal{R}^N$.
4. Eigenvectors can be sorted by their number of zero crossings.
5. The eigenvector with k zero-crossings has the eigenvalue of $e^{-j\frac{\pi k}{2}}$ under DFT operation.

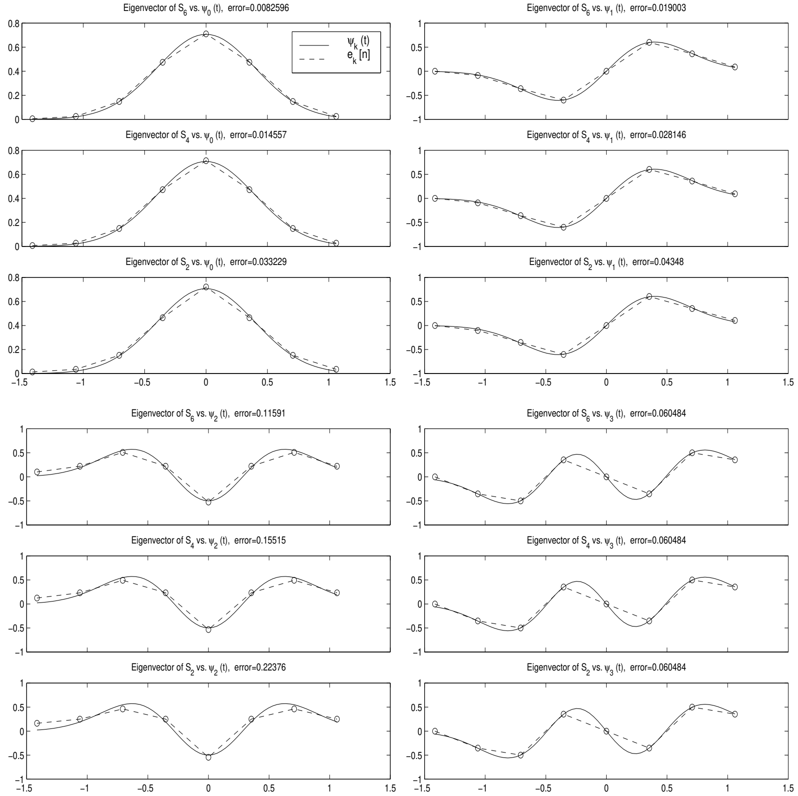
Additionally the relation between the eigenvalues and the number of zero-crossings of each eigenvector is in agreement with the eigenvalue multiplicity of the DFT. That is, for the even dimensional DFT matrices ($N = 2k$), there exists no eigenvector with $2k-1$ zero-crossings and the eigenvalue of the DFT matrix corresponding to the vector with $2k-1$ zero-crossings is also skipped. The relation also holds for odd dimensional DFT matrices.

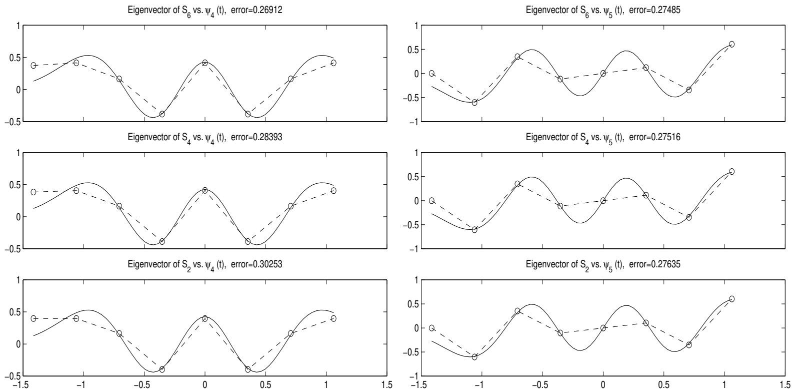
Taking into account the analogy presented above, we propose these vectors as discrete counterparts of the Hermite-Gaussians functions. We believe that these vectors will also be shown to satisfy further discrete counterparts of many properties of continuous Hermite-Gaussian functions such as recurrence relations, generating function etc. This is an area for further research.

5.5 Summary

We have determined "better" approximations for Hermite-Gaussians in this chapter and compared the approximations with the Hermite-Gaussians. In the next chapter we will define the discrete FrFT by the eigenvectors found in this chapter.

Figure 5.1: Hermite-Gaussians and eigenvectors of $\mathbf{S}_{2k}$ matrices, $N = 8$.





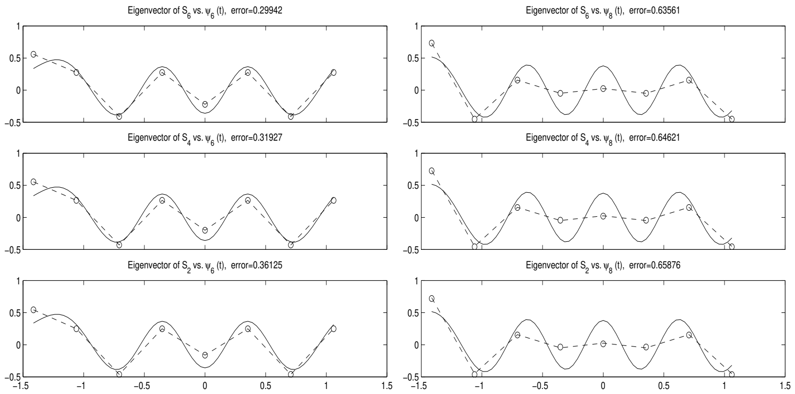
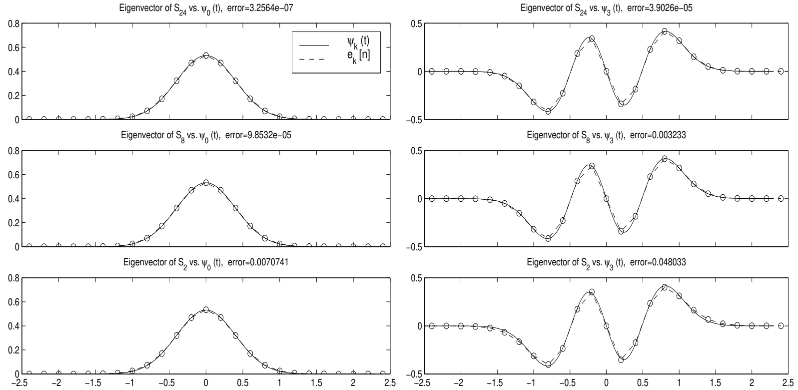
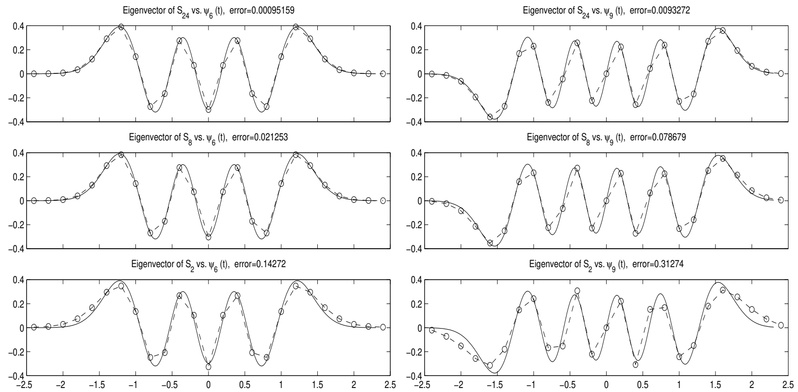
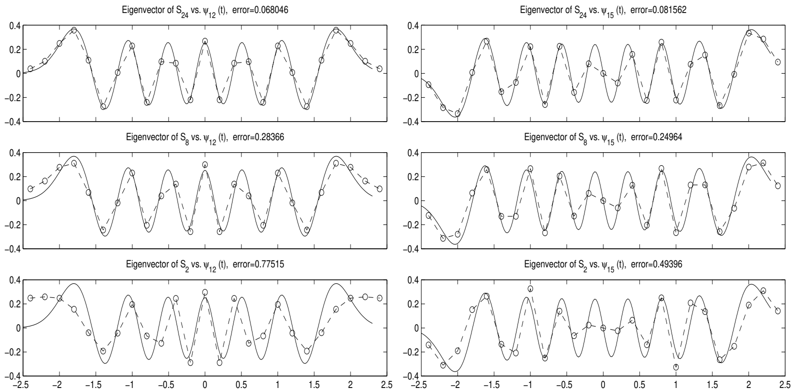


Figure 5.2: Hermite-Gaussians and eigenvectors of $\mathbf{S}_{2k}$ matrices, $N = 25$.







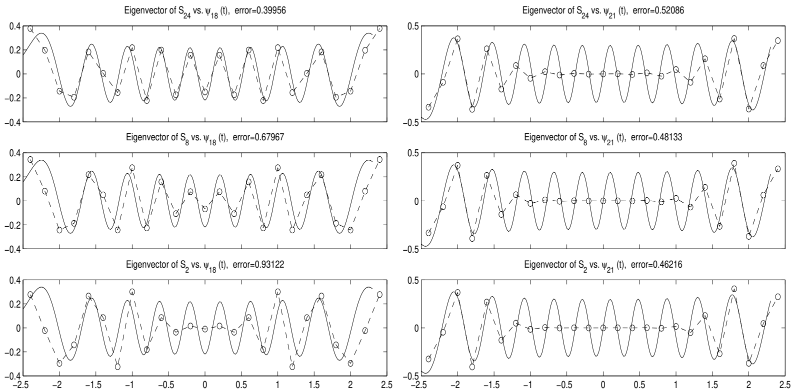
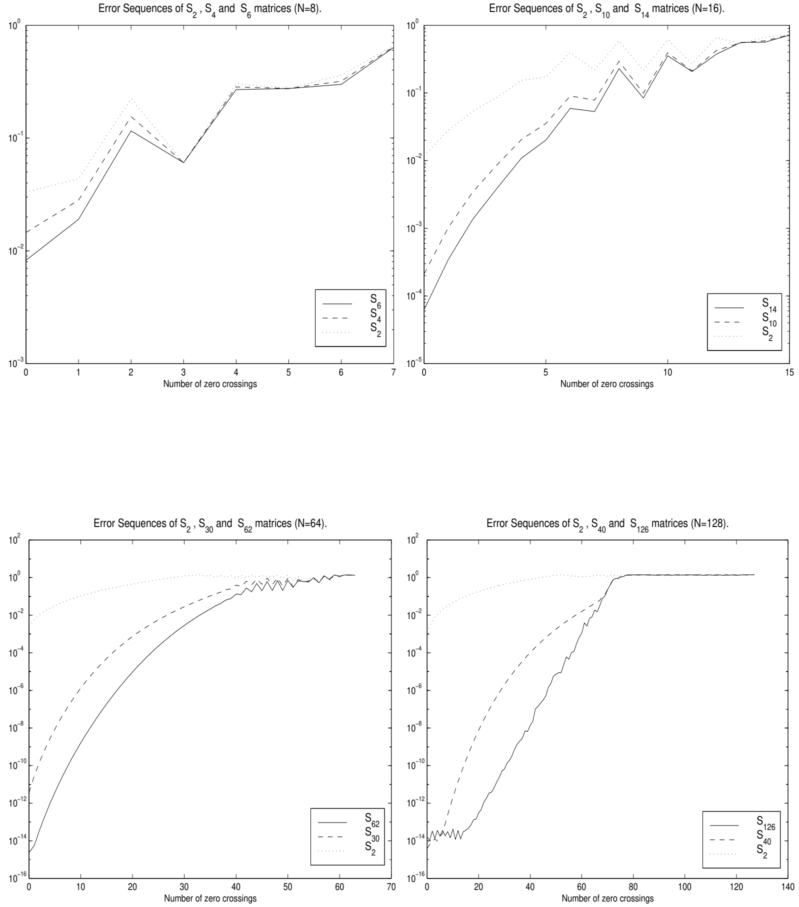


Figure 5.3: Error sequences for $\{8, 16, 64, 128\}$ dimensional $\mathbf{S}_{2k}$ matrices.



Chapter 6

Discrete Fractional Fourier Transform

In this chapter, we will propose a definition for the discrete FrFT satisfying all the requirements and derive some of the properties of the discrete FrFT proposed.

6.1 Definition of the discrete FrFT

We will define the discrete FrFT using spectral expansion of the kernel. The requirements for the discrete FrFT are:

1. Unitarity.

2. Additivity of orders.

3. Reduction to DFT.

4. Correspondence with FrFT.

We have shown in Chapter 2, that any definition, including distinct definitions, satisfying the first three requirements can be defined by spectral expansion. To satisfy the last requirement we set up a similar path in discrete time to the path that had led to the definition of the Hermite-Gaussians. Throughout the Chapters 3 to 5, we have shown that there exists no ambiguity in finding the eigenvectors of DFT through $\mathbf{S}_{2k}$ matrices and it is also shown that the eigenvectors found from $\mathbf{S}_{2k}$ matrices tend to samples of Hermite-Gaussians as N increases. As a result, the eigenvectors and the eigenvalues of the discrete transform coincides with the continuous transform as N tends to infinity, meaning that discrete transform evolves into continuous one as N increases.

6.1.1 Definition

Let $\widetilde{\psi}_k$ be the eigenvector of DFT with k zero-crossings found from $\mathbf{S}_{2k}$ matrix. For N odd, discrete FrFT is defined by

$$K_a(n_a, n) = \sum_{k=0}^{N-1} \widetilde{\psi}_k[n_a] \, e^{-j\frac{\pi a k}{2}} \, \widetilde{\psi}_k[n] \tag{6.1}$$

For N even,

$$K_a(n_a, n) = \sum_{k=0, k\neq(N-1)}^{N} \widetilde{\psi}_k[n_a] \, e^{-j\frac{\pi a k}{2}} \, \widetilde{\psi}_k[n] \tag{6.2}$$

where $K_a(n_a, n)$ denotes the kernel of the discrete FrFT matrix. The lower summation index in (6.2) indicates that there exists no eigenvector with $N-1$ zero-crossings when N is even which can be checked from Section 4.5. The discrete FrFT matrix generation procedure is summarized in Algorithm 1.

Example 1
The 0.5th order discrete FrFT matrix for N=4 is given by

$$\mathbf{F}^{0.5} = \begin{bmatrix} 0.70-0.25i & 0.35+0.25i & 0.25i & 0.35+0.25i \\ 0.35+0.25i & 0.35-0.60i & 0.35-0.25i & -0.35+0.10i \\ 0.25i & 0.35-0.25i & -0.70-0.25i & 0.35-0.25i \\ 0.35+0.25i & -0.35+0.10i & 0.35-0.25i & 0.35-0.60i \end{bmatrix}$$

One can check that $\mathbf{F}^{0.5}$ is unitary, that is $\mathbf{F}^{0.5}(\mathbf{F}^{0.5})^H = \mathbf{I}$ and $\mathbf{F}^{0.5}\mathbf{F}^{0.5} = \mathbf{F}^1$ where $\mathbf{F}^1$ is the 4×4 DFT matrix.

Algorithm 1 Discrete FrFT Matrix Generation Algorithm.

1 Generate $\mathbf{S}_{2k}$, $\mathbf{P}$ matrices.
2 Generate $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$ matrices.
3 Find eigenvectors and eigenvalues of $\mathbf{E_{vn}}$ and $\mathbf{O_{dd}}$.
4 Sort eigenvectors of $\mathbf{E_{vn}}$ ($\mathbf{O_{dd}}$) in the descending order of eigenvalues of $\mathbf{E_{vn}}$ ($\mathbf{O_{dd}}$).
5 Let the sorted eigenvectors be denoted by $\vec{e}_k$ ($\vec{o}_k$) where $0 \le k \le \dim\{\mathbf{E_{vn}}\}$ ($0 \le k \le \dim\{\mathbf{O_{dd}}\}$).
6 Let $\widetilde{\psi}_{2k}[n] = \mathbf{P}\,[\,\vec{e}_k^{\,T} \mid 0 \ldots 0\,]^T$.
7 Let $\widetilde{\psi}_{2k+1}[n] = \mathbf{P}\,[\,0 \ldots 0 \mid \vec{o}_k^{\,T}\,]^T$.
8 $\mathbf{F}^a[m,n] = \sum_{k=\mathcal{M}} \widetilde{\psi}_k[m]\, e^{-j\frac{\pi}{2}ka}\, \widetilde{\psi}_k[n]$, $\mathcal{M} = \{0, \ldots, N-2, (N-(N)_2)\}$

6.2 Properties of discrete FrFT

Some properties of discrete FrFT can be stated as follows:[1]

1. Linearity. Linearity is trivially seen from the matrix form definition of the transform.

2. Unitarity. By construction, discrete FrFT is unitary. One can also state the unitarity of transform as existence of Parseval's relation for all fractional domains, that is

$$\sum_{n=0}^{N-1} |f_{a1}[n]|^2 = \sum_{n=0}^{N-1} |f_{a2}[n]|^2$$

3. Index Additivity. By construction, discrete FrFT satisfies index additivity, that is

$$\mathbf{F}^{a1} f_{a2}[n] = f_{a1+a2}[n]$$

4. Reduction to DFT. By construction, discrete FrFT reduces to Identity, DFT, Parity and Inverse DFT matrices at the orders $a = \{0, 1, 2, 3\}$ respectively.

[1]As in continuous case, the ath order discrete FrFT of the $f[n]$ is denoted by $f_a[n]$.

5. $f[-n] \leftrightarrow f_a[-n]$. Lets call $f^{rev}[n] = f[-n]$.

$$\begin{aligned}
f_a^{rev}[n] &= \sum_l \left(\sum_k \widetilde{\psi}_k[n] \lambda_k^a \widetilde{\psi}_k[l] \right) f[-l] \\
&= \sum_k \widetilde{\psi}_k[n] \lambda_k^a \sum_l \widetilde{\psi}_k[l] f[-l] \\
&= \sum_k \widetilde{\psi}_k[n] \lambda_k^a \sum_l (-1)^k \widetilde{\psi}_k[l] f[l] \\
&= \sum_{k,l} \widetilde{\psi}_k[-n] \lambda_k^a \widetilde{\psi}_k[l] f[l] \\
&= f_a[-n]
\end{aligned}$$

6. $f^*[n] \leftrightarrow f^*_{-a}[n]$. Lets call $f^{conj}[n] = f^*[n]$.

$$\begin{aligned}
f_a^{conj}[n] &= \left(\left(\sum_{k,l} \widetilde{\psi}_k[n] \lambda_k^a \widetilde{\psi}_k[l] f^*[l] \right)^* \right)^* \\
&= (f_{-a}[n])^*
\end{aligned}$$

7. $\mathbf{E_{vn}}\{f[n]\} \leftrightarrow \mathbf{E_{vn}}\{f_a[n]\}$. Lets call $f^{evn}[n] = \frac{f[n]+f[-n]}{2}$, where indices are in modulo N.

$$\begin{aligned}
f_a^{evn}[n] &= \sum_{k,l} \widetilde{\psi}_k[n] \lambda_k^a \widetilde{\psi}_k[l] f^{evn}[l] \\
&= \sum_{k=even,l} \widetilde{\psi}_k[n] \lambda_k^a \widetilde{\psi}_k[l] f^{evn}[l] \\
&= \mathbf{E_{vn}}\{f_a[n]\}
\end{aligned}$$

8. $\mathbf{O_{dd}}\{f[n]\} \leftrightarrow \mathbf{O_{dd}}\{f_a[n]\}$. Proof is similar to the one of property 7.

Unfortunately shift, modulation and other simple properties of the DFT can not be analytically derived for the discrete FrFT, without closed form definition of the kernel of the transform.

6.3 Comparison with Continuous FrFT

As we have discussed earlier, by construction of the discrete FrFT, we expect the kernel of the discrete FrFT to approach the continuous kernel. In this section we will compare the discrete and continuous FrFT for the two sample input functions namely “rectangle” and “triangle” functions.

The continuous FrFT of the sample functions are calculated by numerical integration using “quad8” function of the MATLAB with the tolerance of $1e-3$ and the discrete FrFT is calculated by the definition given in this chapter, using the samples of the continuous functions at N points around zero, taken with the sampling rate $h = \frac{1}{\sqrt{N}}$.

Figure 6.1 shows the discrete FrFT and continuous FrFT output, when input is “rectangle” function for $N = 32$ and for the orders $a = \{0.05, 0.25, 0.5, 0.75\}$. The discrete FrFT matrix used in this example is generated from eigenvectors of $\mathbf{S}_{30}$.

Figure 6.2 shows the outputs for $N = 64$, where FrFT matrix is derived from eigenvectors of $\mathbf{S}_{62}$.

Figure 6.3 and 6.4 shows a similar comparison for “triangle” function.

6.4 Summary

In this chapter, a definition for the discrete FrFT is given using discrete counterpart of Hermite-Gaussian functions. Properties of the discrete FrFT and the numerical comparison results with the continuous FrFT is presented.

Figure 6.1: Discrete and Continuous FrFT of rect function, $N = 32$.

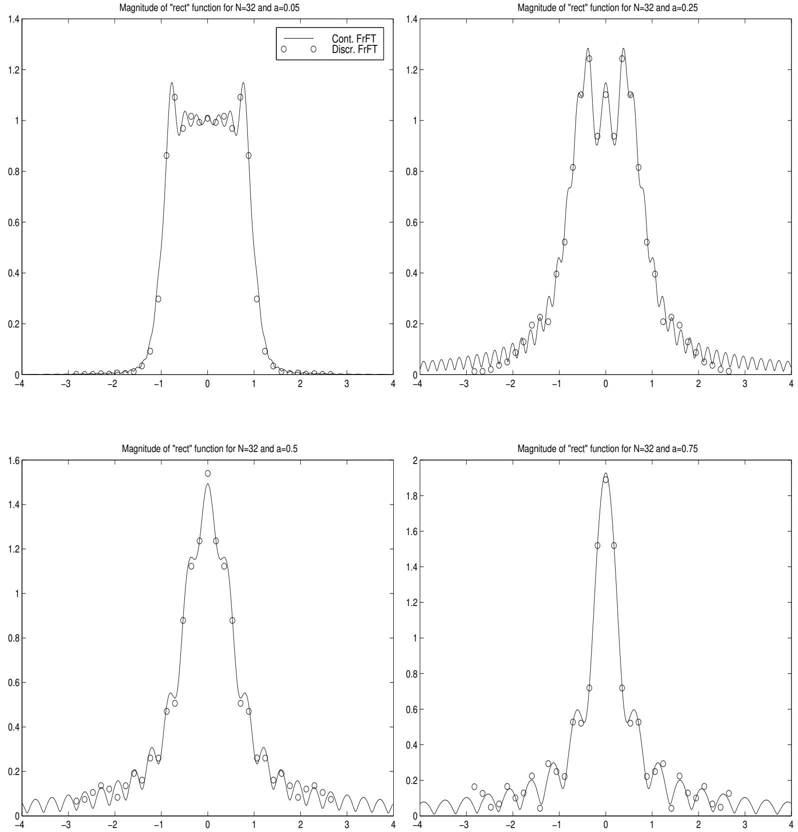


Figure 6.2: Discrete and Continuous FrFT of rect function, $N = 64$.

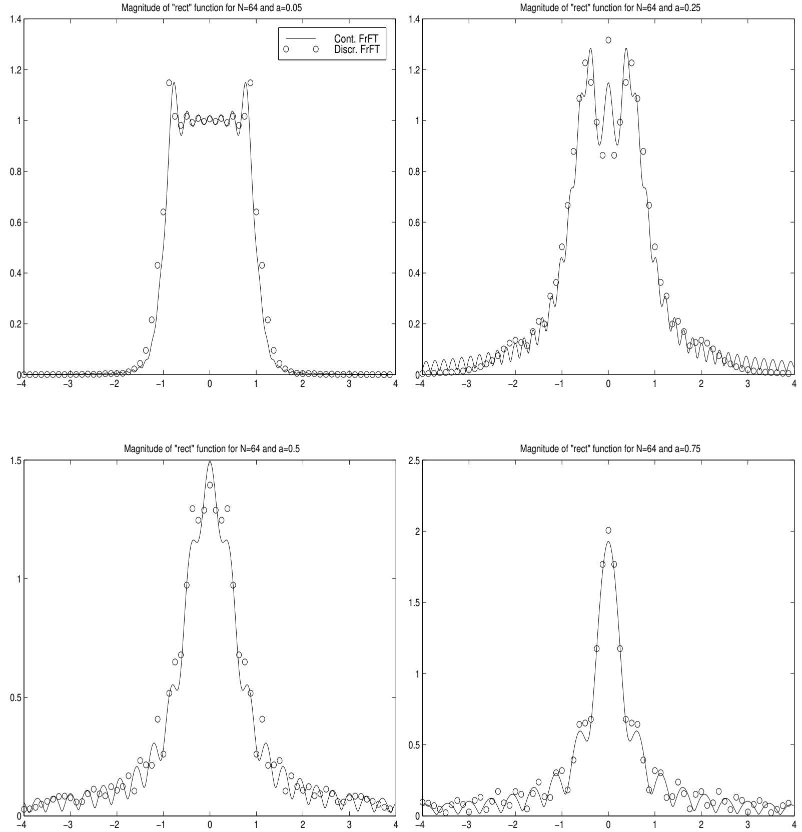


Figure 6.3: Discrete and Continuous FrFT of triangle function, $N = 32$.

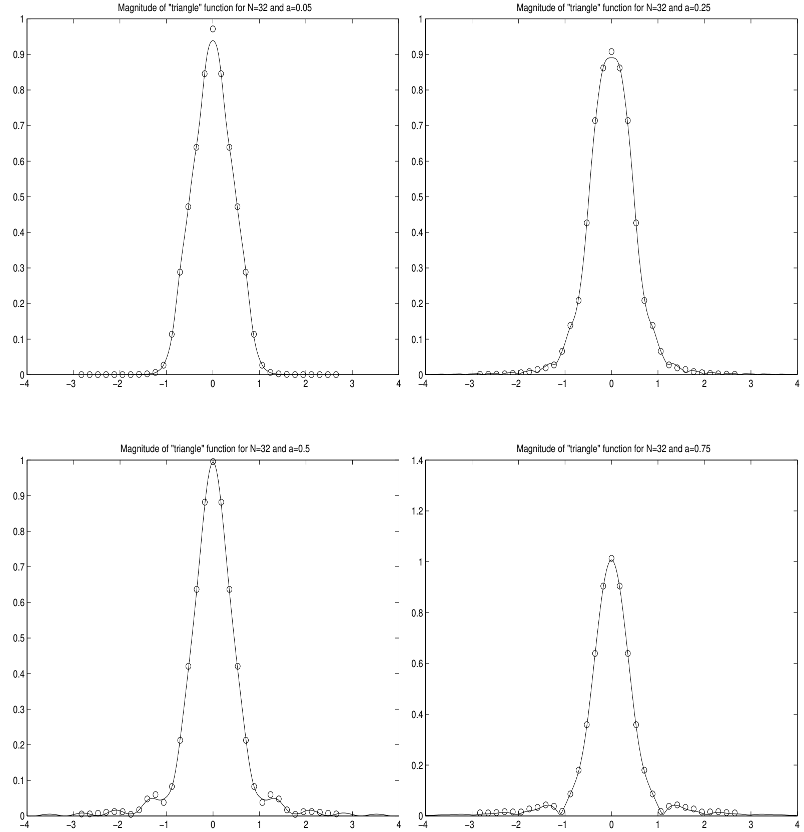
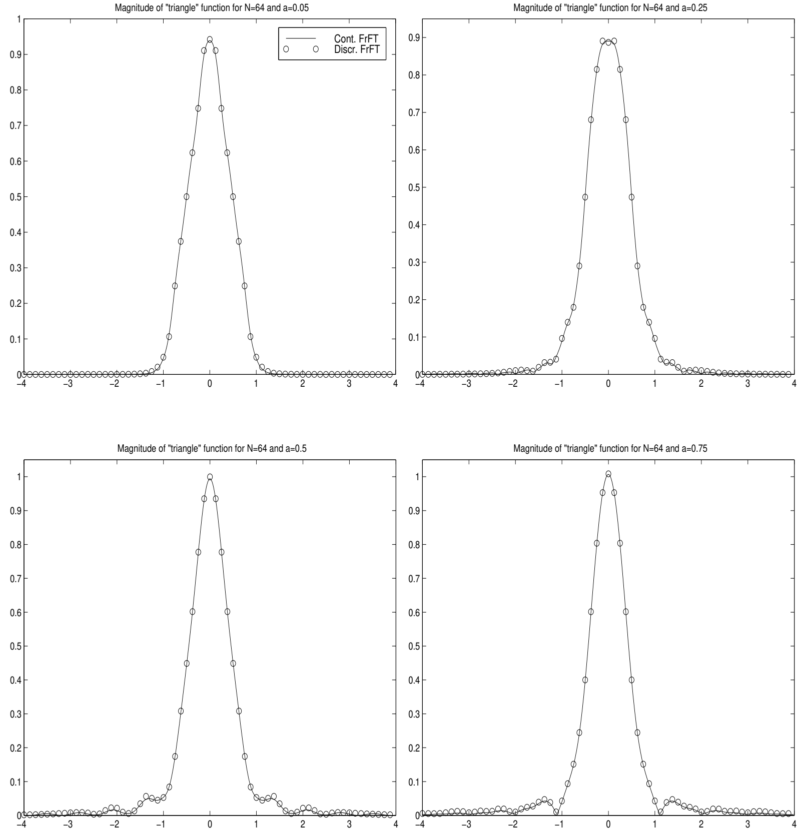


Figure 6.4: Discrete and Continuous FrFT of triangle function, $N = 64$.



Chapter 7

Conclusions

In this work, we present a definition for the discrete fractional Fourier Transform. The definition proposed satisfies the requirements of unitarity, index additivity and reduction to DFT. We have shown that as the dimension of the discrete FrFT matrix increases, discrete FrFT matrix tends to the kernel of the continuous FrFT.

First of all, reader may have noted that we have given a number of matrices that can be derived from different $\mathbf{S}_{2k}$ matrices, for the definition of the discrete FrFT. It is clear that each of these matrices differs by the order of approximation to the Hermite-Gaussians. Therefore, if one is only interested in calculation of the continuous FrFT of the input, it would be more appropriate to use the highest order transformation matrix, since eigenvectors of this matrix are the best approximations to the Hermite-Gaussians. If the signal to be transformed has no continuous origin, then any one of the discrete FrFT matrices can be utilized, since they all have the same footing with respect to each other.

Our derivation of the discrete transform by spectral expansion closely resembles the continuous definition given in [2, 3]. In these papers, authors define the continuous FrFT by spectral expansion using Hermite-Gaussians. It has been later discovered that in [5], Namias derived the integral kernel of

FrFT from spectral expansion using some properties of the Hermite-Gaussians (mainly generating function of Hermite-Gaussians). The existence of such relations for Hermite-Gaussians is not surprising, since Hermite-Gaussians are well studied under the general setting of orthogonal functions and therefore there exists many results for Hermite-Gaussians in the literature. To reach the closed form definition for the discrete FrFT that we propose, we need similar relations for the eigenvectors of $\mathbf{S}_{2k}$ matrices; but apart from the difference equation satisfied by these vectors, we do not have any results at our hand. Further work on the $\mathbf{S}_{2k}$ may yield results leading to the closed form definition of discrete FrFT.

In [22], six different, but equivalent, definitions for the continuous FrFT are given and the relations between these definitions are discussed. Although it is possible to derive any definition from any other, authors show that the derivation of some of the properties of the FrFT can be eased by the utilization of a specific definition. The definition, we have given for the discrete FrFT, is analogous to "Definition B" in [22]. Providing other definitions for the discrete FrFT analogous to the ones of the continuous transform can be useful to clarify the definition of the discrete transform. The most accessible definitions among these six definitions, in the sense of integrating into a definition for the discrete FrFT, are the definition E and definition F. Both of these definitions use the continuous $\mathbf{S}$ operator, which is actually Hamiltonian of the harmonic oscillator system, to define the FrFT. As the reader may have noted, in this work, we define discrete $\mathbf{S}_{2k}$ operators, that are tending to the continuous Hamiltonian. Similar to the continuous-time operator, the discrete-time operator is invariant under DFT operation, which is the counterpart of the Fourier Transform in discrete time. Furthermore, since $\mathbf{S}_{2k}$ operator has orthogonal eigenvectors, we propose $\mathbf{S}_{2k}$ operator as the discrete counterpart of the harmonic oscillator Hamiltonian. Returning back to the definitions E and F; since these definitions utilize continuous Hamiltonian, one may think to replace continuous Hamiltonian with the discrete one to get the definition for the discrete FrFT. Unfortunately continuous Hamiltonian has a spectrum (eigenvalues) that is uniform and this uniform nature of eigenvalues becomes important for the definitions E and F (see [22]). But the discrete Hamiltonian has eigenvalues that are not uniform in any sense, leading to problems with the

definition of discrete FrFT. Further work may ease the problem of the definition of the discrete FrFT by providing analogous definitions to the definitions in [22].

If we examine these definitions more closely, definition E defines the FrFT as the solution of an differential equation analogous to the time-dependent Schroedinger's equation. If we write the discrete counterpart of this equation using the same method as in Chapters 4 and 5, the resulting difference equation can be taken as the definition of the discrete FrFT.

In definition F, the FrFT is defined as $\mathbf{F}^a = e^{-j\frac{\pi}{2}\mathbf{S}}$ where $\mathbf{S}$ is the continuous Hamiltonian given in Chapter 3. One may be able to define discrete FrFT, by replacing $\mathbf{S}$ by $\mathbf{S}_{2k}$.

Further work on FrFT can also be based on prolate spheroidal functions. These functions are the eigenfunctions of the Fourier Transform in a finite interval. It is clear that these functions evolve into Hermite-Gaussians as interval enlarges. It may be of interest to study discretization problem through these functions. For details about prolate spheroidal functions, one can consult [39, 58].

Another avenue to be explored for the discrete FrFT can be group theory. We know that the continuous FrFT generalizes the Fourier Transform to a general family of transforms which satisfies the group properties (see [22]). If the fractional Fourier group can be defined from integer order Fourier transform group using abstract group theoretical concepts, one may be able to apply the same method of fractionalization to the DFT operation. The key idea would be to find the counterpart of DFT matrix in the same sense that the discrete rotation group is the counterpart of the 90 degree and multiples continuous rotation group.[1]

An important part of the theory that remains to be clarified is the relationship between discrete Wigner distribution and the discrete FrFT, analogous to the projection and rotation properties (see [24]). Conversely, the discrete FrFT defined may find application in *defining* the discrete Wigner distribution.

[1] [35] and references therein can be a good starting point for the group theoretical method.

A fundamental aspect of the Fourier Transform is the duality of periodicity (finite extent) in one domain and discreteness (sampling) in the other domain. Generalization of this notion to fractional domains remains an intriguing aspect of the theory which remains to be clarified. This aspect is also closely related to the relation of the transform to the discrete Wigner distribution.

A fast algorithm for calculating the FrFT has been given in [25], but it would be more satisfying to start from the discrete FrFT matrix and present a fast algorithm which multiplies precisely by this matrix. We believe that such an algorithm will emerge soon.

Another interesting topic is the clarification of how the discrete transform is precisely related to the continuous transform beyond stating that "discrete FrFT is an approximation to FrFT". Clearly this problem constitutes the generalization of the Poisson's theorem to FrFT.

In this thesis, we have not derived all possible properties of the discrete Hermite-Gaussian vectors as well as the discrete FrFT. We also believe that these gaps can be easily filled.

Lastly, the definition of the discrete FrFT given in this work, can be viewed as an infinite sequence of unitary matrices, which includes Identity and DFT matrices. A certain member of this sequence, DFT, and its derivatives, DCT, Hartley, etc. are proved to be successful in many applications such as coding, filtering. Generalization of these DFT based transforms, can also lead to interesting results which may be of importance.

APPENDIX A

Poisson's Theorem

We know that N point DFT is an approximation to the Discrete Time Fourier Transform (DTFT) [59], it is also well known that as N increases approximation of DFT also improves (windowing effect). If discrete signal is formed by sampling the continuous signal with a rate higher than the Nyquist rate, then DTFT of the discrete signal is the same as the Fourier Transform of the continuous signal. As a result, one can conclude that, as N increases N point DFT output becomes a better approximation of the samples of the continuous time Fourier Transform of the signal. Poisson's theorem [39], not only justifies the above comments, but also gives a quantitative expression for the approximation.

Poisson's Theorem [39]:

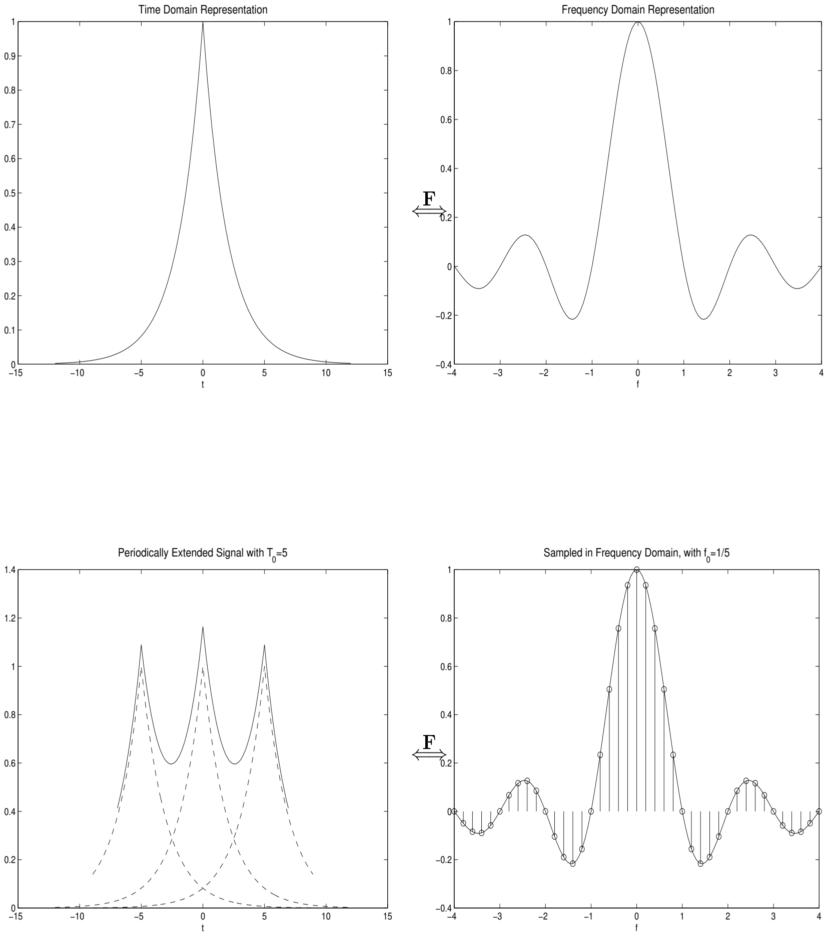
$$f_{alias}(t) = \sum_{n=-\infty}^{\infty} f(t+nT_0) \qquad F_{alias}(f) = \sum_{n=-\infty}^{\infty} F(f+nf_1)$$

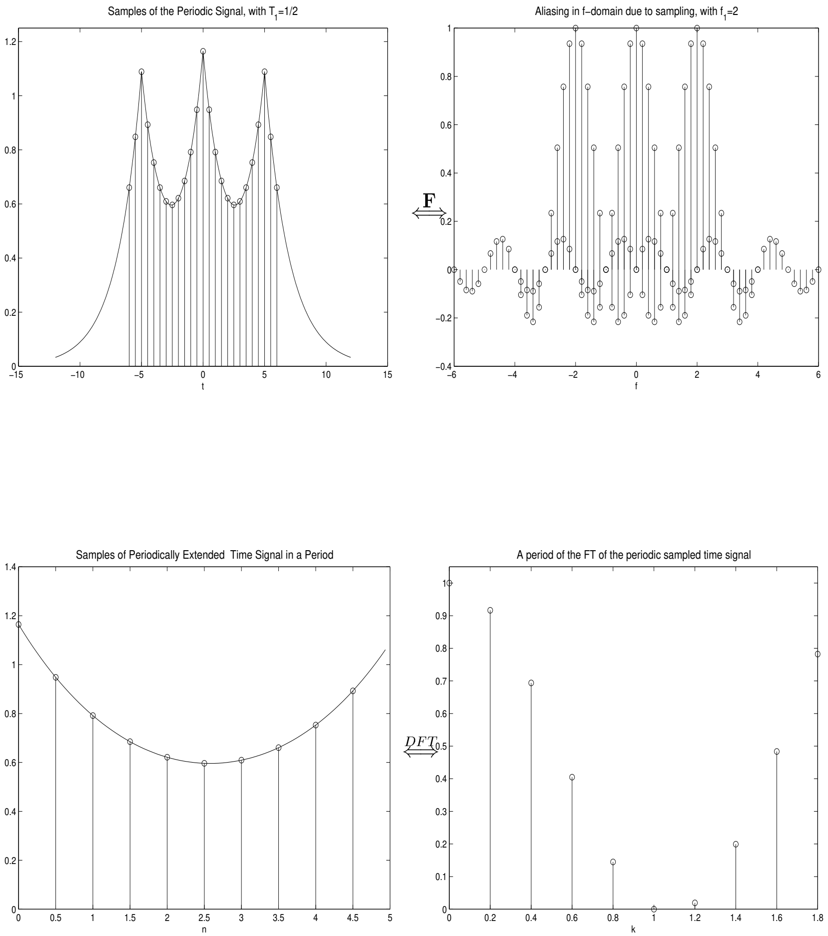
$$T_1 = \frac{T_0}{N} \qquad f_1 = Nf_0 = \frac{1}{T_1}$$

$$f_{alias}(mT_1) = \frac{1}{T}\sum_{n=0}^{N-1} F_{alias}(nf_0)e^{j\frac{2\pi mn}{N}} \tag{A.1}$$

If we explain the theorem in words, let $f(t)$, $F(f)$ be the signal and its Fourier Transform respectively, then $f_{alias}(t)$ is periodic extension of $f(t)$ with period T_0, Reader should note that extending a signal periodically in time, corresponds to sampling in frequency (Fourier series); therefore Fourier Transform of $f_{alias}(t)$ is the samples of $F(f)$ taken with the rate $\frac{1}{T_0}$. If the periodic function, $f_{alias}(t)$, is sampled at N points in a period T_0; that is sampled with T_1, function becomes periodically extended in frequency with period f_1 (aliasing of the sampling theorem). One should note that after extension and sampling in both domains; we reach a discrete periodic sequence in time and frequency domain. Poisson's theorem says that the mapping between the discrete time sequences generated by extending and sampling operations in both domains is DFT. It is clear that as N increases samples in frequency will be "less" aliased, leading to a better approximation. We see from Poisson's theorem that DFT of the discrete signal has a correspondence with Fourier Transform of the signal. Other features of DFT such as unitarity, fast implementation makes this transform not only an approximation to Fourier Transform, but turns it into a well defined working transform for discrete signals.

Figure A.1: Poisson's Theorem By Pictures, $T_0 = 5$, $T_1 = 1/2$.





APPENDIX B

Eigenvector Sets of DFT

The following Matlab lines generates a different orthogonal eigenvector set of DFT matrix each time it is run.

```
N=7;F=dftmtx(N)/sqrt(N); [ Evec,Eval] =eig(F);
Eval=diag(Eval); Evec=real(Evec);
   index1=find((abs(Eval-1))<1e-3);
   indexm1=find((abs(Eval+1))<1e-3);
   indexi=find((abs(Eval-i))<1e-3);
   indexmi=find((abs(Eval+i))<1e-3);
Evec(:,index1) =Evec(:,index1)*rand(length(index1));
Evec(:,indexm1)=Evec(:,indexm1)*rand(length(indexm1));
Evec(:,indexi) =Evec(:,indexi)*rand(length(indexi));
Evec(:,indexmi)=Evec(:,indexmi)*rand(length(indexmi));
   Evec=orth(Evec);
```

APPENDIX C

Approximations for $\mathbf{D}^2$

In the first section we review calculus of finite differences and in the following section we derive $O(h^{2k})$ approximations for $\mathbf{D}^2$.

C.1 Calculus of Finite Differences

In this section, we will present the required background for the method of finite differences. Reader can refer to the excellent book of Hildebrand [56] for more details.

We will first define finite differencing operators.
Forward difference operator,

$$\triangle f(t) = f(t+h) - f(t) \tag{A.1}$$

Backward difference operator,

$$\bigtriangledown f(t) = f(t) - f(t-h) \tag{A.2}$$

Central Difference operator,

$$\delta f(t) = f(t+\frac{h}{2}) - f(t-\frac{h}{2}) \tag{A.3}$$

and the shift operator,

$$\mathbf{E}f(t) = f(t+h) \tag{A.4}$$

Operators defined above are closely related with each other, that is

$$\triangle = \mathbf{E} - 1 \quad , \quad \bigtriangledown = 1 - \mathbf{E}^{-1} \quad , \quad \delta = \mathbf{E}^{\frac{1}{2}} - \mathbf{E}^{-\frac{1}{2}} \tag{A.5}$$

$$\delta = \mathbf{E}^{-\frac{1}{2}}\triangle \quad , \quad \delta = \mathbf{E}^{\frac{1}{2}}\bigtriangledown \tag{A.6}$$

One easily verifies using the relations above that,

$$\triangle\bigtriangledown = \bigtriangledown\triangle = \delta^2 \tag{A.7}$$

A number of identities can be drawn using $\mathbf{E} = \triangle + 1$. An important one is

$$\begin{aligned} \mathbf{E}^r f_k &= (1+\triangle)^r f_k \\ f_{k+r} &= f_k + \frac{r}{1!}\triangle f_k + \frac{r(r-1)}{2!}\triangle^2 f_k + \cdots + \triangle^r f_k \end{aligned} \tag{A.8}$$

where $f_k = f(t+kh)$. Replacing k by zero and r by s in (A.8),

$$f_s = f_0 + \frac{s}{1!}\triangle f_0 + \frac{s(s-1)}{2!}\triangle^2 f_0 + \cdots + \frac{s(s-1)\cdots(s-r+1)}{s!}\triangle^s f_0 \tag{A.9}$$

When s is an non-negative integer (A.9) reduces to (A.8), otherwise it can be observed that right hand side of (A.9) does not terminate. One way of viewing (A.9) is, interpreting s as a continuous variable, leading to a polynomial of degree n on the right hand side (assume s equals a positive integer n), interpolating the sequence of values $\{f_0, f_1, \ldots, f_n\}$. The same equation can also be viewed as the discrete analog of the Taylor series expansion of a discrete function, where derivatives in Taylor series are replaced with $\triangle$, and s^k terms are replaced by factorial polynomials, $s^{(k)}$

$$s^{(k)} = s(s-1)(s-2)\ldots(s-k+1) \tag{A.10}$$

One should note that $s^{(k)}$ is also a kth degree polynomial as s^k, but it has also roots than zero such as $\{1, 2, \ldots, k-1\}$. As a result Taylor series in discrete domain, known as Newton's formula can be expressed as

$$f_s = \sum_{k=0}^{\infty} \frac{s^{(k)}}{k!}\triangle^k f_0 \tag{A.11}$$

To strengthen the relation between the factorial polynomials and forward difference operator, we also show that $\triangle s^{(n)} = ns^{(n-1)}$.

$$\begin{aligned}\triangle s^{(n)} &= (s+1)^{(n)} - (s)^{(n)} \\ &= (s+1)s^{(n-1)} - s^{(n-1)}(s-n+1) \\ &= ns^{(n-1)} \qquad \text{(A.12)}\end{aligned}$$

C.2 Approximations for $\mathbf{D}^2$

In this section, we will prove the relation between $(\widetilde{\mathbf{D}^2})_{2k}$ and $\mathbf{D}^2$. We will prove the results for the finite degree polynomials of arbitrary degrees, but results of this section are also valid for infinite polynomials under some convergence conditions.

In the last section, we have shown that $\triangle k^{(n)} = nk^{(n-1)}$, where $k^{(n)}$ is the nth degree factorial polynomial. Using the relation $\delta^2 = \mathbf{E}^{-1}\triangle^2$, one can write

$$\begin{aligned}\delta^2 k^{(n)} &= \mathbf{E}^{-1}\triangle^2 k^{(n)} \\ &= (n)(n-1)(k-1)^{(n-2)} \qquad \text{(A.13)}\end{aligned}$$

(A.13) can also be shown by using definitions of δ^2 and $k^{(n)}$.

We will need another result that will be utilized later, which can be stated as

$$\delta^2\{k(k+n)^{(2n+1)}\} = (2n+2)(2n+1)k(k+n-1)^{(2n-1)} \qquad \text{(A.14)}$$

and proved by

$$\begin{aligned}\triangle\{k(k+n)^{(2n+1)}\} &= (k+1)(k+n+1)^{(2n+1)} - k(k+n)^{(2n)} \\ &= k(2n+1)(k+n)^{(2n)} + (k+n+1)^{(2n+1)} \\ \triangle^2\{k(k+n)^{(2n+1)}\} &= k(2n+1)(2n)(k+n)^{(2n-1)} + 2(2n+1)(k+n+1)^{(2n)} \\ &= 2(2n+1)\{nk+k+n+1\}(k+n)^{(2n-1)} \qquad \text{(A.15)}\end{aligned}$$

finally by using $\delta^2 = \mathbf{E}^{-1}\triangle^2$,

$$\delta^2\{k(k+n)^{(2n+1)}\} = (2n+2)(2n+1)k\,(k+n-1)^{(2n-1)} \qquad \text{(A.16)}$$

We will first rewrite an arbitrary polynomial in terms of factorial polynomials and operator δ^2. One can check that a relation between factorial polynomials and operator $\triangle$ is given the previous section, where relation is named as discrete counterpart of the Taylor series.

Assume $f(k)$ is an arbitrary polynomial, we will write the polynomial $f(k)$ as $f(k) = f_e(k) + f_o(k)$, where $f_e(k)$ is the polynomial of only even powered terms, and $f_o(k)$ consists of odd powered terms. One can easily see that $f_e(k)$ and $f_o(k)$ is an even, odd function respectively.

We will expand $f_e(k)$ and $f_o(k)$ polynomials, or equivalently $f(k)$, by some other even and odd polynomials. The polynomials that will be used in the expansion are $k(k+l-1)^{(2l-1)}$ and $(k+l-1)^{(2l-1)}$ where the first one is an even polynomial, the second one is odd. Evenness of $k(k+l-1)^{(2l-1)}$ can be seen from the definition $k^{(n)}$, that is

$$\begin{aligned}k(k+l-1)^{(2l-1)} &= k(k+l-1)\ldots(k+1)\,k\,(k-1)\ldots(k-(l-1)) \qquad \text{(A.17)} \\ &= k^2(k^2-1)(k^2-4)\ldots(k^2-(l-1)^2) \qquad \text{(A.18)}\end{aligned}$$

We claim that $f(k_0+k)$ can be written as

$$\begin{aligned}f(k_0+k) &= \left(\sum_{l=1}^{n} a_{2l}\delta^{2l}\left(f(k_0)\right)\, k(k+l-1)^{(2l-1)} + a_0 f(k_0)\right) + \ldots \\ &\left(\sum_{l=1}^{n} a_{2l-1}\delta^{2l-2}\left(f(k_0+1)\right)\,(k+l-1)^{(2l-1)}\right) \qquad \text{(A.19)}\end{aligned}$$

where $\{a_0,\ldots,a_k\}$ are determined such that equality in (A.19) is satisfied for all k values. By brute force methods, one can expand both $f(k_0+k)$ and $k^{(n)}$ polynomials and compare the coefficients of the each k^n term, to determine a_n coefficients. Easier method of finding coefficients can be accomplished by inserting certain values for k, and solving the linear systems of equations for a_n coefficients. If one inserts $k=0$ in (A.19), one gets

$$f(k_0) = a_0 f(k_0) \qquad \text{(A.20)}$$

which implies $a_0 = 1$. To find other a_n coefficients, assume that (A.19) is satisfied and operate from left by δ^2, that is

$$\begin{aligned}\delta^2 f(k_0+k) &= a_2\delta^2 f(k_0)2.1 + \left(\sum_{l=2}^{n}(2l)(2l-1)a_{2l}\delta^{2l}\left(f(k_0)\right)\,k(k+l-2)^{(2l-3)}\right) + \ldots \\ &\left(\sum_{l=2}^{n}(2l-1)(2l-2)a_{2l-1}\delta^{2l-2}\left(f(k_0+1)\right)\,(k+l-2)^{(2l-3)}\right) \qquad \text{(A.21)}\end{aligned}$$

where (A.16) and (A.13) is used at the left of the (A.21). Now by fixing $k=0$ in (A.21), we directly find a_2 which is $a_2 = 1/2$. Similarly it is easily seen that $a_{2n} = \frac{1}{(2n)!}$.

To find odd a_{2k-1} coefficients, we will fix $k=1$ in (A.19), leading to

$$f(k_0+1) \quad = \quad a_0\delta^2 f(k_0) + a_2\delta^4 f(k_0) + a_1 f(k_0+1) \qquad \text{(A.22)}$$

Since we have found a_{2k} coefficients, one can determine a_1 from (A.22). To find a_3 one only needs to insert $k=1$ in (A.21).

One can observe the similarity of the method for finding a_n coefficients and the method for partial fractional expansion of the ratio of two polynomials.

It is now clear that arbitrary degree polynomials can be expanded at point k_0, that is

$$\begin{aligned} f(k_0+k) \quad &= \quad \left(\sum_{l=1}^{n} \frac{1}{(2l)!}\delta^{2l}\left(f(k_0)\right)\, k(k+l-1)^{(2l-1)} + f(k_0)\right) + \ldots \\ &\quad \left(\sum_{l=1}^{n} a_{2l-1}\delta^{2l-2}\left(f(k_0+1)\right)\, (k+l-1)^{(2l-1)}\right) \qquad \text{(A.23)} \end{aligned}$$

We will now establish the result of this section by differentiating (A.23) twice, that is

$$\begin{aligned} \frac{d^2}{d\,k^2} f(k_0+k) \quad &= \quad \left(\sum_{l=1}^{n} \frac{1}{(2l)!}\delta^{2l}\left(f(k_0)\right)\, \frac{d^2}{d\,k^2} k(k+l-1)^{(2l-1)}\right) + \ldots \\ &\quad \left(\sum_{l=1}^{n} a_{2l-1}\delta^{2l-2}\left(f(k_0+1)\right)\, \frac{d^2}{d\,k^2}(k+l-1)^{(2l-1)}\right) \text{(A.24)} \end{aligned}$$

If we fix $k=0$ in (A.24),

$$\begin{aligned} f''(k_0) = &\left(\sum_{l=1}^{n} \frac{1}{(2l)!}\delta^{2l}\left(f(k_0)\right)\, \left(2.\mathrm{Coef}\{k^2 \text{ of } k(k+l-1)^{(2l-1)}\}\right)\right) + \ldots \\ &\left(\sum_{l=1}^{n} a_{2l-1}\delta^{2l-2}\left(f(k_0+1)\right)\, \left(2.\mathrm{Coef}\{k^2 \text{ of } (k+l-1)^{(2l-1)}\}\right)\right) \text{(A.25)} \end{aligned}$$

One can find the k^2 coefficient of $k(k+l-1)^{(2l-1)}$ and $(k+l-1)^{(2l-1)}$ as $(-1)^{l-1}[(l-1)!]^2$ and zero respectively. Then one can reduce (A.25) to

$$f''(k_0) \quad = \quad 2\sum_{l=1}^{n}(-1)^{l-1}\frac{[(l-1)!]^2}{(2l)!}\delta^{2l}\left(f(k_0)\right) \qquad \text{(A.26)}$$

which completes the proof of this section.

First few terms of the summation in (A.26) is,

$$f''(k_0) \quad = \quad \delta^2 f(k_0) - \frac{1}{12}\delta^4 f(k_0) + \frac{1}{90}\delta^6 f(k_0) + \cdots \qquad \text{(A.27)}$$

which is in agreement with the special case cited in [56, page 124]. Lastly it can be shown that the result in (A.26) remains the same; when difference operators δ and $k^{(n)}$ are generalized for arbitrary shifts of h units, that is when δ is replaced with $\{\frac{\delta^2_h}{h^2}\}$ which is defined in (5.3) and $k^{(n)}$ is replaced with polynomials of type $\{k(k-h)(k-2h)\ldots(k-(n-1)h)\}$. For the notational simplicity we present for the special case of $h=1$.

The relation (A.25) is exact for the polynomials of finite degrees, leading to exact expression for the second derivative. In general one can express $\mathbf{D}^2$ operator for infinite polynomials as

$$\mathbf{D}^2 f(t) \quad = \quad 2\sum_{l=1}^{\infty}(-1)^{l-1}\frac{[(l-1)!]^2}{(2l)!}\delta^{2n}\left(f(t)\right) \qquad \text{(A.28)}$$

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