THE DISCRETE FRACTIONAL FOURIER TRANSFORM

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ABSTRACT

We propose and consolidate a definition of the discrete fractional Fourier transform which generalizes the discrete Fourier transform (DFT) in the same sense that the continuous fractional Fourier transform (FRT) generalizes the continuous ordinary Fourier Transform. This definition is based on a particular set of eigenvectors of the DFT which constitutes the discrete counterpart of the set of Hermite-Gaussian functions. The fact that this definition satisfies all the desirable properties expected of the discrete FRT, supports our confidence that it will be accepted as the definitive definition of this transform.

1. INTRODUCTION

In recent years, the fractional Fourier transform (FRT) has attracted a considerable amount of attention, resulting in many applications in the areas of optics and signal processing. However, a satisfactory definition of the discrete FRT, consistent with the continuous transform, has been lacking. In this paper, our aim is to propose (following Pei and Yeh [1]) and consolidate a definition which has the same relation with the continuous FRT, as the DFT has with the ordinary continuous Fourier transform. This definition has the following properties, which may be posed as requirements to be satisfied by a legitimate discrete-input/discrete-output FRT:

1. Unitarity.
2. Index additivity.
3. Reduction to the DFT when the order is equal to unity.
4. Approximation of the continuous FRT.

A comprehensive introduction to the FRT and historical references may be found in [2]. The transform has become popular in the optics and signal processing communities following the works of Ozaktas and Mendlovic [3, 4], Lohmann [5] and Almeida [6]. Some of the applications explored include optimal filtering in fractional Fourier domains [7], cost-efficient linear system synthesis and filtering [8, 9] and time-frequency analysis [6, 2]. Further references may be found in [2].

A fast $O(N \log N)$ algorithm for digitally computing the continuous fractional Fourier transform integral has been given in [10]. This method maps the N samples of the original function to the N samples of the transform. Whereas this mapping is very satisfactory in terms of accuracy, the $N \times N$ matrix underlying this mapping is not *exactly* unitary and does not *exactly* satisfy the index additivity property. This makes it unsuitable for a self-consistent a priori definition of the discrete transform.

Several publications proposing a definition for the discrete FRT have appeared, but none of these papers satisfies all of the above requirements [10, 11, 12, 13, 14]. The definition proposed in this paper was first suggested by Pei and Yeh [1]. They suggest defining the discrete FRT in terms of a particular set of eigenvectors (previously discussed in [14]) which they claim to be the discrete analogs of the Hermite-Gaussian functions. They justify their claims by numerical observations and simulations. In the present paper we provide an analytical development of Pei's claims with the aim of consolidating the definition of the discrete FRT.

2. PRELIMINARIES

2.1. Continuous Fractional Fourier Transform

The continuous FRT can be defined through its integral kernel:

$$\{\mathcal{F}^a f\}(t_a) = \int_{-\infty}^{\infty} K_a(t_a, t) f(t) dt \tag{1}$$

where $K_a(t_a,t) = K_\phi e^{j\pi(t_a^2 \cot\phi - 2t_a t \csc\phi + t^2 \cot\phi)}$ and $\phi = a\frac{\pi}{2}$. The kernel is known to have the following spectral expansion [15]:

$$K_a(t_a, t) = \sum_{k=0}^{\infty} \psi_k(t_a)\, e^{-j\frac{\pi}{2}ka}\, \psi_k(t) \tag{2}$$

where $\psi_k(t)$ denotes the kth Hermite-Gaussian function and t_a denotes the variable in the ath order *fractional Fourier domain* [4]. Here $\exp(-j\pi ka/2)$ is the ath power of the eigenvalue $\lambda_k = \exp(-j\pi k/2)$ of the ordinary Fourier transform. When $a = 1$, the FRT reduces to the ordinary Fourier transform. As a approaches zero or integer multiples of ± 2, the kernel approaches $\delta(t_a - t)$ and $\delta(t_a + t)$ respectively [16]. The most important properties of the FRT are **1.** Unitarity, **2.** Index additivity: $\mathcal{F}^{a_1}\mathcal{F}^{a_2} = \mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_1+a_2}$, **3.** Reduction to the ordinary Fourier transform when $a = 1$.

We will define the discrete FRT through a discrete analog of (2). Therefore, we will first discuss the Hermite-Gaussian functions in some detail.

2.2. The Hermite-Gaussian functions

The kth order Hermite-Gaussian function is defined as ($k = 0, 1, \ldots$)

$$\psi_k(t) = \frac{2^{1/4}}{\sqrt{2^k k!}} H_k\left(\sqrt{2\pi}t\right) e^{-\pi t^2} \tag{3}$$

where H_k is the kth Hermite polynomial having k real zeros. The Hermite-Gaussians form a complete and orthonormal set in $\mathcal{L}_2$. The Hermite-Gaussian functions are well known to be the eigenfunctions of the Fourier transform, as will also be seen below.

We begin with the defining differential equation of the Hermite-Gaussians :

$$\frac{d^2 f(t)}{dt^2} - 4\pi^2 t^2 f(t) = \lambda f(t) \tag{4}$$

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It can be shown that the Hermite-Gaussian functions are the unique finite energy eigensolutions of (4) [17]. We can express the left hand side of (4) in operator notation as

$$\left(\mathcal{D}^2 + \mathcal{F}\mathcal{D}^2\mathcal{F}^{-1}\right) f(t) \quad = \quad \lambda f(t) \qquad (5)$$

where $\mathcal{D} = \frac{d}{dt}$ and $\mathcal{F}$ denote differentiation and the ordinary Fourier transformation respectively. The operator $(\mathcal{D}^2 + \mathcal{F}\mathcal{D}^2\mathcal{F}^{-1})$ is the Hamiltonian associated with the harmonic oscillator [18]. Here we will denote this operator by $\mathcal{S}$ and thus write (5) as $\mathcal{S}f(t) = \lambda f(t)$.

A theorem of commuting operators will be used to show that the Hermite-Gaussian functions, which are eigenfunctions of $\mathcal{S}$, are also eigenfunctions of $\mathcal{F}$ [19, page 52].

Theorem 1 *If $\mathcal{A}$ and $\mathcal{B}$ commute (i.e. $\mathcal{AB} = \mathcal{BA}$), there exists a common eigenvector set between $\mathcal{A}$ and $\mathcal{B}$.*

We can see that $\mathcal{F}$and $\mathcal{S}$ commute as follows:

$$\begin{aligned} \mathcal{FS} = \mathcal{F}\mathcal{D}^2 + \mathcal{F}^2\mathcal{D}^2\mathcal{F}^{-1} &= \mathcal{F}\mathcal{D}^2 + \mathcal{F}^2\mathcal{D}^2\mathcal{F}^{-2}\mathcal{F} \\ &= \mathcal{F}\mathcal{D}^2 + \mathcal{D}^2\mathcal{F} = \mathcal{SF} \qquad (6) \end{aligned}$$

where we used $\mathcal{F}^2\mathcal{D}^2\mathcal{F}^{-2} = \mathcal{D}^2$ which follows from $\mathcal{F}^2 = \mathcal{F}^{-2} = \mathcal{R}$, $\mathcal{R}f(t) = f(-t)$. Using theorem 1 and the fact that Hermite-Gaussian functions are the unique eigenfunctions of $\mathcal{S}$, we conclude that they are also the eigenfunctions of $\mathcal{F}$.

3. THE DISCRETE FRACTIONAL FOURIER TRANSFORM

We will first show that the first three requirements are automatically satisfied when we define the transform through a spectral expansion analogous to (2). Assuming $p_k[n]$ to be an arbitrary *orthonormal* eigenvector set of the $N \times N$ DFT matrix and λ_k the associated eigenvalues, the discrete analog of (2) is

$$\mathbf{F}^a[m,n] \quad = \quad \sum_{k=0}^{N-1} p_k[m]\ (\lambda_k)^a\ p_k[n] \qquad (7)$$

The matrix $\mathbf{F}^a$ is unitary since the eigenvalues λ_k have unit magnitude (since the DFT matrix is unitary). Reduction to the DFT when $a = 1$ follows trivially, since when $a = 1$ (7) reduces to the spectral expansion of the ordinary DFT matrix. Index additivity can likewise be easily demonstrated by multiplying the matrices $\mathbf{F}^{a_1}$ and $\mathbf{F}^{a_2}$ and using the orthonormality of the $p_k[n]$.

Before we continue, we note that there are two ambiguities which must be resolved in (7). The first concerns the eigenstructure of the DFT. Since the DFT matrix has only 4 distinct eigenvalues ($\lambda_k \in \{1, -1, j, -j\}$) [20], the eigenvalues are degenerate so that the eigenvector set is not unique. In the continuous case, this ambiguity is resolved by choosing the common eigenfunction set of the commuting operators $\mathcal{S}$ and $\mathcal{F}$ which are the Hermite-Gaussian functions. Analogously in discrete case, we will resolve this ambiguity by choosing the common eigenvector set of the DFT matrix and the discrete matrix analog of $\mathcal{S}$. These eigenvectors may be considered to be the discrete counterparts of the Hermite-Gaussian functions. They will be derived in the next section.

The second ambiguity is associated with the fractional power appearing in (7), since the fractional power operation is not single valued. This ambiguity will again be resolved by analogy with the continuous case given in (2), i.e. we take $\lambda_k^a = \exp(-i\pi ka/2)$.

3.1. Discrete Hermite-Gaussians

We will define the discrete Hermite-Gaussians as eigensolutions of a difference equation which is analogous to the defining differential equation (4) of the continuous Hermite-Gaussian functions. First we define the second difference operator $\widetilde{\mathcal{D}}^2$

$$\frac{\widetilde{\mathcal{D}}^2}{h^2} f(t) \quad = \quad \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} \qquad (8)$$

$\widetilde{\mathcal{D}}^2$ serves as an approximation to d^2/dt^2. $\widetilde{\mathcal{D}}^2$ can be related to $\mathcal{D}^2$ as

$$\frac{\widetilde{\mathcal{D}}^2}{h^2} \quad = \quad \frac{e^{h\mathcal{D}} - 2 + e^{-h\mathcal{D}}}{h^2} = \mathcal{D}^2 + \underbrace{\frac{2h^2}{4!}\mathcal{D}^4 + \cdots}_{O(h^2)} \qquad (9)$$

where we we have expressed the shift operator in hyperdifferential form: $f(t+h) = e^{h\mathcal{D}} f(t)$ [18, 21].

Now, we consider the factor $\mathcal{F}\widetilde{\mathcal{D}}^2\mathcal{F}^{-1}$ appearing in $\mathcal{S}$ which can be evaluated as

$$\mathcal{F}\widetilde{\mathcal{D}}^2\mathcal{F}^{-1} = \frac{2\left(\cos(2\pi ht) - 1\right)}{h^2} = -4\pi^2 t^2 + O(h^2) \qquad (10)$$

where we used the fact that $\mathcal{F}e^{h\mathcal{D}}\mathcal{F}^{-1} = e^{j2\pi ht}$, which is nothing but a statement of the shift property of the ordinary Fourier transform.

Now, we replace $\mathcal{D}^2$ in (5) with $\frac{\widetilde{\mathcal{D}}^2}{h^2}$ to obtain an approximation of $\mathcal{S}$, which we refer to as $\widetilde{\mathcal{S}}$:

$$\begin{aligned} \widetilde{\mathcal{S}} = \frac{\widetilde{\mathcal{D}}^2}{h^2} + \mathcal{F}\frac{\widetilde{\mathcal{D}}^2}{h^2}\mathcal{F}^{-1} &= \frac{\widetilde{\mathcal{D}}^2}{h^2} + \frac{2\left(\cos(2\pi ht) - 1\right)}{h^2} \\ &= \mathcal{D}^2 - 4\pi t^2 + O(h^2) \qquad (11) \end{aligned}$$

If we explicitly write the difference equation $\widetilde{\mathcal{S}}f(t) = \lambda f(t)$, we obtain

$$f(t+h) + f(t-h) + 2\left(\cos(2\pi ht) - 2\right) f(t) = h^2 \lambda f(t) \quad (12)$$

We convert this equation to a finite difference equation by setting $t = nh$ [21] with $h = \frac{1}{\sqrt{N}}$:

$$f_{n+1} + f_{n-1} + 2\left(\cos\left(\frac{2\pi}{N}n\right) - 2\right) f_n = \lambda f_n \qquad (13)$$

where $f_n = f(nh)$. One should note that the coefficients of (13) are periodic with N, implying the existence of periodic vectors as eigensolutions of this difference equation [22]. Concentrating on the period defined by $0 \le n \le N-1$, we obtain a system of equations of the form $\mathbf{Sf} = \lambda\mathbf{f}$.

$$\mathbf{S} = \begin{bmatrix} C_0 & 1 & 0 & \dots & 1 \\ 1 & C_1 & 1 & \dots & 0 \\ 0 & 1 & C_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & C_{N-1} \end{bmatrix} \qquad (14)$$

where $C_n = 2(\cos(\frac{2\pi}{N}n) - 2)$. This symmetric matrix commutes with the DFT matrix ensuring the existence of common eigenvectors. Furthermore this common set can be shown to be *unique* and *orthogonal* [22]. These facts will be substantiated below. It is this eigenvector set $\mathbf{u}_k$ which will be taken as the discrete counterpart of continuous Hermite-Gaussians.

Theorem 2 *The matrix* **S** *and the DFT matrix (***F***) commute.*

Proof: **S** can be written as $\mathbf{S} = \mathbf{A} + \mathbf{B}$, where **A** is the circulant matrix corresponding to the system whose impulse response is $h[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$, and **B** is the diagonal matrix defined as $\mathbf{B} = \mathbf{FAF}^{-1}$. It can also be shown that $\mathbf{A} = \mathbf{FBF}^{-1}$ since $h[n]$ is an even function. Then $\mathbf{FSF}^{-1} = \mathbf{F}(\mathbf{A}+\mathbf{B})\mathbf{F}^{-1} = \mathbf{B}+\mathbf{A} = \mathbf{S}$. ■

We will now show that the common eigenvector set is unique. First recall that eigenvectors of the DFT matrix are either even or odd sequences [20]. Thus the common eigenvector set should also consist of even or odd vectors. We will introduce a matrix **P** which decomposes an arbitrary vector $f[n]$ into its even and odd components. This matrix maps the even part of $f[n]$ to the first $\lfloor (N/2+1) \rfloor$ components and the odd part to the remaining components.[1] For example, matrix **P** for $N = 5$ is

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \quad (15)$$

and satisfies $\mathbf{P} = \mathbf{P}^T = \mathbf{P}^{-1}$. The similarity transformation $\mathbf{PSP}^{-1}$ can be written as

$$\mathbf{PSP}^{-1} = \begin{bmatrix} \mathbf{Ev} & \mathbf{0} \\ \mathbf{0} & \mathbf{Od} \end{bmatrix} \quad (16)$$

where the **Ev** and **Od** matrices are *symmetric tri-diagonal* matrices with the dimensions $\lfloor (N/2+1) \rfloor$ and $\lfloor (N-1)/2 \rfloor$ respectively. Since tri-diagonal matrices have distinct eigenvalues [19], the **Ev** and **Od** matrices have a unique set of eigenvectors. When the eigenvectors of **Ev** and **Od** are zero-padded and multiplied with **P**, we get the *unique* even-odd orthogonal eigenvector set of **S**. That is, an even eigenvector of **S** is obtained as $\mathbf{P}[\mathbf{e}_k{}^T \mid 0 \ldots 0]^T$ where $\mathbf{e}_k$ is an eigenvector of **Ev**. Similarly, an odd eigenvector set is obtained from the eigenvectors of **Od** as $\mathbf{P}[0 \ldots 0 \mid \mathbf{o}_k{}^T]^T$. Thus we have shown how to obtain the unique common eigenvector set.

We will now show how to order this vector set in a manner consistent with the ordering of the continuous Hermite-Gaussians. The kth Hermite-Gaussian has k zeros (3). Analogously, we will order the eigenvectors of **S** in terms of the number of their zero-crossings.[2] In counting the number of zero-crossings of the periodic sequence $f[n]$ (with period N), we count the number of zeros in the period $n = \{0, \ldots, N-1\}$, also including the zero-crossing at the boundary, i.e. $f[N-1]f[N] = f[N-1]f[0] < 0$ [22].

Since directly counting the number of zero-crossings of each vector is numerically problematic (due to the very small magnitude of certain components), we will employ the following indirect method: As discussed before, the common eigenvectors of **S** and the DFT can be derived from eigenvectors of the tri-diagonal **Ev** and **Od** matrices. An explicit expression for the eigenvectors of tri-diagonal matrices are given in [19, page 316]. Combining this expression with the Sturm sequence theorem [19, page 300], one can show that the eigenvectors of the **Ev** or **Od** matrices with the highest eigenvalue has no zero-crossings, the eigenvector with the second highest eigenvalue has one zero-crossing, and so on.

[1] $\lfloor x \rfloor$ is the greatest integer less than or equal to the argument.

[2] The vector $f[n]$ has a zero-crossing at n if $f[n]f[n+1] < 0$.

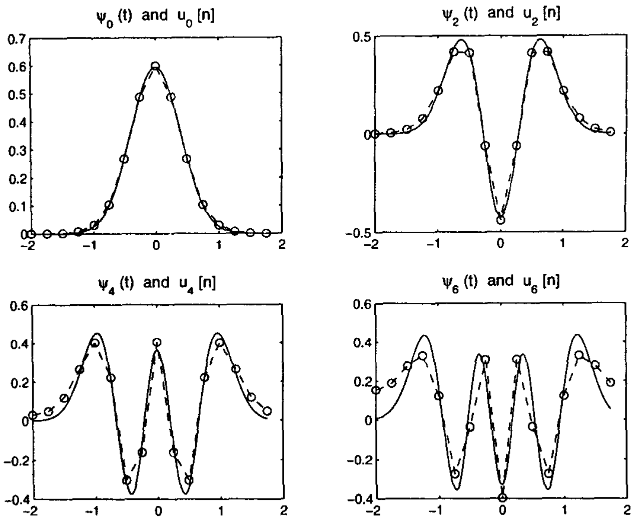

Figure 1: Comparison of the $\{0, 2, 4, 6\}$th Hermite-Gaussian functions with the corresponding eigenvectors of 16×16 the **S** matrix.

Therefore one can show that the **Ev** and **Od** matrices have eigenvectors whose numbers of zero-crossings range from 0 to $\lfloor N/2 \rfloor$ and to $\lfloor (N-3)/2 \rfloor$ respectively.

Since the even and odd eigenvectors of **S** are derived from the zero padded eigenvectors of the **Ev** and **Od** matrices, one can show that after zero padding and multiplication with **P**, the eigenvector of **Ev** with k zero-crossings yields the even eigenvector of **S** with $2k$ ($0 \le k \le \lfloor N/2 \rfloor$) zero-crossings and the eigenvector of **Od** with k zero-crossings yields the odd eigenvector of **S** with $2k+1$ ($0 \le k \le \lfloor (N-3)/2 \rfloor$) zero-crossings. This procedure not only enables us to accurately determine the number of zero-crossings, but also demonstrates that each of the eigenvectors of **S** has a different number of zero-crossings so that the ordering in terms of zero-crossings is unambiguous.

In Fig. 1, eigenvectors of **S** are compared with the corresponding Hermite-Gaussian functions.

3.2. Discrete Fractional Fourier Transform

The definition of the discrete FRT can now be given as

$$\mathbf{F}^a[m,n] = \sum_{k=0, k \neq (N-1+(N)_2)}^{N} u_k[m]\, e^{-j\frac{\pi}{2}ka}\, u_k[n] \quad (17)$$

where $u_k[n]$ is the eigenvector of **S** with k zero-crossings and $(N)_2 \equiv N \bmod 2$. The peculiar range of summation is due to the fact that there does not exist an eigenvector with $N-1$ or N zero-crossings when N is even or odd respectively. The overall procedure is summarized in Table 1.

Lastly, we present a numerical comparison of the discrete and continuous transforms for a sample input function in Fig. 2.

4. CONCLUSIONS

We have presented a definition of the discrete FRT which exactly satisfies the essential operational properties of the continuous frac-

Table 1: Generation of Matrix $\mathbf{F}^a$

1	Generate matrices $\mathbf{S}$ and $\mathbf{P}$.
2	Generate the $\mathbf{Ev}$ and $\mathbf{Od}$ matrices from (16).
3	Find the eigenvectors/eigenvalues of $\mathbf{Ev}$ and $\mathbf{Od}$.
4	Sort the eigenvectors of $\mathbf{Ev}$ ($\mathbf{Od}$) in the descending order of eigenvalues of $\mathbf{Ev}$ ($\mathbf{Od}$) and denote the sorted eigenvectors as $\mathbf{e}_k$ ($\mathbf{o}_k$).
5	Let $\mathbf{u}_{2k}[n] = \mathbf{P}\,[\,\mathbf{e}_k{}^T \mid 0 \ldots 0\,]^T$. Let $\mathbf{u}_{2k+1}[n] = \mathbf{P}\,[\,0 \ldots 0 \mid \mathbf{o}_k{}^T\,]^T$.
6	Define $\mathbf{F}^a[m,n] = \sum_{k \in \mathcal{M}} u_k[m]\, e^{-j\frac{\pi}{2}ka}\, u_k[n]$, $\mathcal{M} = \{0, \ldots, N-2, (N-(N)_2)\}$

tional Fourier transform. This definition sets the stage for a self-consistent discrete theory of the fractional Fourier transformation and makes possible a priori discrete formulations in applications.

As a by-product, we obtained the discrete counterparts of the Hermite-Gaussian functions. We believe that the discrete counterparts of the multitude of operational properties for the Hermite-Gaussian functions, such as recurrence relations, differentiation properties, etc. can be derived by methods similar to those in Section 3.

We already mentioned that the $O(N \log N)$ algorithm presented in [10] can be utilized for fast computation in most applications. However, it would be preferable to have a fast algorithm which exactly computes the product of the fractional Fourier transform matrix defined here, with an arbitrary vector.

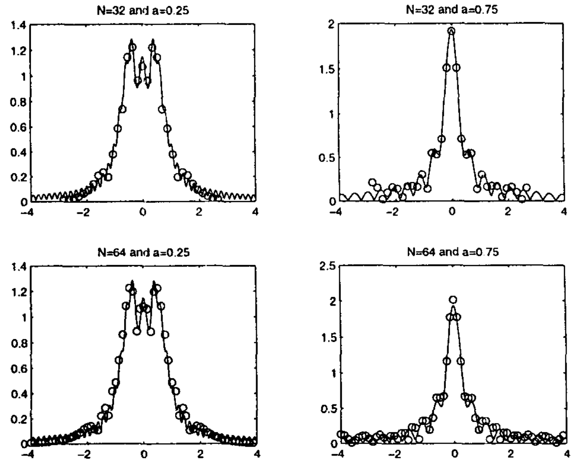

Figure 2: Magnitude of the continuous (solid curve) and discrete (circles) FRT of the function $z(t) = 1$ if $|t| < 1, 0$ otherwise.

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